Electrodynamics A (PHY 5346) Fall 2016 Solutions

Set 6:

(19) Exercise: Lorentz transformation for the field of a point charge.

In K' the field is

$$\vec{E}^{\,\prime} = \frac{q\,\left(x'\,\hat{e}^1 + y'\,\hat{e}^2 + z'\,\hat{e}^3\right)}{\left(x'^{\,2} + y'^{\,2} + z'^{\,2}\right)^{3/2}}$$

This field transforms from K' to K with the inverse transformations

$$\vec{E} = \gamma\,\vec{E}^{\,\prime} - \frac{\gamma^2}{\gamma+1}\,\vec{\beta}\,\left(\vec{\beta}\cdot\vec{E}^{\,\prime}\right) \quad \text{and} \quad \vec{B} = \gamma\,\vec{\beta}\times\vec{E}^{\,\prime}\,.$$

Therefore, with $\vec{\beta} = \beta\,\hat{e}^1$ we get

$$\begin{aligned}
E^1 &= \gamma\,E'^{\,1} - \frac{\gamma^2}{\gamma+1}\,\beta^2\,E'^{\,1} \;=\; \frac{\gamma^2 + \gamma - \gamma^2\beta^2}{\gamma+1}\,E'^{\,1} \\
&= \frac{\cosh^2\zeta + \gamma - \sinh^2\zeta}{\gamma+1}\,E'^{\,1} \;=\; \frac{\gamma+1}{\gamma+1}\,E'^{\,1} = E'^{\,1}\,, \\
E^2 &= \gamma\,E'^{\,2}\,, \quad E^3 \;=\; \gamma\,E'^{\,3}\,, \\
B^1 &= 0\,, \quad B^2 = -\gamma\,\beta\,E'^{\,3}\,, \quad B^3 = +\gamma\,\beta\,E'^{\,2}\,.
\end{aligned}$$

It remains to express the arguments of the $\vec{E}^{\,\prime}$ components in the coordinates of K:

$$\begin{aligned}
c\,t' &= +\gamma\,c\,t - \gamma\,\beta\,x\,, \\
x' &= -\gamma\,\beta\,c\,t + \gamma\,x\,, \\
y' &= y\,, \quad z' \;=\; z\,,
\end{aligned}$$

resulting in

$$
\begin{aligned}
E^1 &= \frac{q\,(-\gamma\,\beta\,c\,t+\gamma\,x)}{\left[(-\gamma\,\beta\,c\,t+\gamma\,x)^2+y^2+z^2\right]^{3/2}}\,,\\
E^2 &= \frac{q\,\gamma\,y}{\left[(-\gamma\,\beta\,c\,t+\gamma\,x)^2+y^2+z^2\right]^{3/2}}\,,\\
E^3 &= \frac{q\,\gamma\,z}{\left[(-\gamma\,\beta\,c\,t+\gamma\,x)^2+y^2+z^2\right]^{3/2}}\,,\\
B^1 &= 0\,,\\
B^2 &= \frac{-q\,\gamma\,\beta\,z}{\left[(-\gamma\,\beta\,c\,t+\gamma\,x)^2+y^2+z^2\right]^{3/2}}\,,\\
B^3 &= \frac{+q\,\gamma\,\beta\,y}{\left[(-\gamma\,\beta\,c\,t+\gamma\,x)^2+y^2+z^2\right]^{3/2}}\,.
\end{aligned}
$$

(20) Classwork: Divergence and Laplace operator.

(a)

$$
\nabla = \sum_{i=1}^{3} \hat{e}^i\,\frac{\partial}{\partial x^i}\,,\quad \nabla\cdot\vec{r} \;=\; \sum_{i=1}^{3}\frac{\partial x^i}{\partial x^i} \;=\; 3\,.
$$

(b)

$$
\nabla = \hat{r}\,\frac{\partial}{\partial r}+\frac{\hat{\theta}}{r}\,\frac{\partial}{\partial\theta}+\frac{\hat{\phi}}{r\sin\theta}\,\frac{\partial}{\partial\phi}\,,
$$

$$
\nabla\cdot\,\vec{r}\,f(r) = f(r)\,\nabla\cdot\vec{r}+\vec{r}\cdot\nabla f = \;3\,f(r)+\vec{r}\cdot\hat{r}\,\frac{df}{dr} = 3\,f(r)+r\,\frac{df}{dr}\,.
$$

(c) Example: Calculate (b) for $f(r) = r^{n-1}$. Solution:

$$
\nabla\cdot\vec{r}\,r^{n-1} = 3\,r^{n-1}+r\,\frac{dr^{n-1}}{d\,r} = 3\,r^{n-1}+(n-1)\,r^{n-1} = (n+2)\,r^{n-1}\,.
$$

(d) Solution for $n=-2$, $r\neq 0$:

$$
\nabla\cdot\vec{E} = \nabla\cdot\frac{q\,\vec{r}}{r^3} = (-2+2)\,\frac{q}{r^3} = 0\,.
$$

(e) Solution for $r\neq 0$:

$$
\nabla^2\,\frac{q}{r} = \nabla\cdot\left(\nabla\,\frac{q}{r}\right) = -\nabla\cdot\left(\frac{q\,\hat{r}}{r^2}\right) = -\nabla\cdot\left(\frac{q\,\vec{r}}{r^3}\right) = 0\,.
$$