Electrodynamics A (PHY 5346) Fall 2016 Solutions

Set 12:

(37) **Expansion up to Cartesian quadrupoles.**

(a) For $|\vec{x}| > |\vec{x}\,'|$ we write

$$\frac{1}{|\vec{x}-\vec{x}\,'|} \;=\; (r^2 - 2\,\vec{x}\cdot\vec{x}\,' + r'^2)^{-1/2} = \\ r^{-1}\,(1 - 2\,\vec{x}\cdot\vec{x}\,'/r^2 + r'^2/r^2)^{-1/2} = r^{-1}\,(1+\epsilon)^{-1/2}$$

with $\epsilon = -2\,\vec{x}\cdot\vec{x}\,'/r^2 + r'^2/r^2$ and $|\epsilon| < 1$. The Taylor expansion of $f(1+\epsilon)$ about $f(1)$ reads

$$f(1+\epsilon) \;=\; f(1) + \epsilon\, f'(1) + \frac{\epsilon}{2}\, f''(1) + \mathcal{O}(\epsilon^3)$$

and in out situation

$$f(1+\epsilon) \;=\; (1+\epsilon)^{-1/2}\,, \quad f'(1+\epsilon) = -\frac{1}{2}\,(1+\epsilon)^{-3/2} \\ \text{and} \quad f''(1+\epsilon) = \frac{3}{4}\,(1+\epsilon)^{-5/2}\,.$$

Therefore, $f(1) = 1$, $f'(1) = -1/2$ and $f''(1) = 3/4$ and up to order ϵ we have

$$\frac{1}{|\vec{x}-\vec{x}\,'|} \;=\; \frac{f}{r} \;=\; \frac{1}{r} + \frac{\vec{x}\cdot\vec{x}\,'}{r^3} - \frac{1}{2}\,\frac{r'^2}{r^3}$$

and integration over the first two terms gives

$$\int d^3x'\,\frac{\rho(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|} \;=\; \frac{q}{r} + \frac{\vec{p}\cdot\vec{x}}{r^3}\,.$$

From the order ϵ^2 we keep only the term $(2\,\vec{x}\cdot\vec{x}\,')^2/r^4$. So we are left with the task to integrate

$$-\frac{1}{2}\,\frac{r'^2}{r^3} + \frac{3}{4}\,\frac{(2\,\vec{x}\cdot\vec{x}\,')^2}{r^4}\,,$$

which results in order of the terms in

$$-\frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} Q_D^{ij}\,\frac{x^i x^j}{r^5} + \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} Q_S^{ij}\,\frac{x^i x^j}{r^5}\,.$$

Note:

$$\sum_{i=1}^{3}\sum_{j=1}^{3} x^i\, Q_D^{ij}\, x^j = r^2 \int d^3x'\, r'^2\; \rho(\vec{x}\,'),$$

$$\int d^3x'\, (\vec{x}\cdot\vec{x}\,')^2\, \rho(\vec{x}\,') = \vec{x}\cdot \int d^3x'\, \vec{x}\,'\, \rho(\vec{x}\,')\, \vec{x}\,'\cdot\vec{x} \;=\; \sum_{i=1}^{3}\sum_{j=1}^{3} x^i\, Q_S^{ij}\, x^j\,.$$

(38) We write the quadrupole moments as

$$Q^{ij} = Q_S^{ij} - Q_D^{ij} \;\text{ with }\; Q_S^{ij} = 3\int d^3x\, x^i\, x^j\, \rho(\vec{x}),\;\; Q_D^{ij} = \int d^3x\, r^2\, \delta^{ij}\, \rho(\vec{x})\,.$$

Results:

$$q_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\int d^3x\,(x-iy)^2\rho(\vec{x}) \;=\; \frac{1}{4}\sqrt{\frac{15}{2\pi}}\int d^3x\,(x^2-ixy-y^2)^2\rho(\vec{x})$$

$$= \frac{1}{12}\sqrt{\frac{15}{2\pi}}\,\left(Q_S^{11} - 2i\,Q^{12} - Q_S^{22}\right) = \frac{1}{12}\sqrt{\frac{15}{2\pi}}\,\left(Q^{11} - 2i\,Q^{12} - Q^{22}\right),$$

$$q_{21} = -\sqrt{\frac{15}{8\pi}}\int d^3x\, z\,(x-iy)\,\rho(\vec{x}) \;=\; -\frac{1}{3}\sqrt{\frac{15}{8\pi}}\,\left(Q^{13} - i\,Q^{23}\right),$$

$$q_{20} = \frac{1}{2}\sqrt{\frac{5}{4\pi}}\int d^3x\,(3z^2-r^2)\,\rho(\vec{x}) \;=\; \frac{1}{2}\sqrt{\frac{5}{4\pi}}\left(Q_S^{33} - Q_D^{33}\right) \;=\; \frac{1}{2}\sqrt{\frac{5}{4\pi}}\,Q^{33}\,.$$

(39) Exercise E.55: Number of Cartesian multipoles.

There are $l+1$ ways to distribute l entries on two histograms. So, for three histograms we have

$$1 + 2 + \cdots + l + (l+1) = (l+1)\,(l+2)/2$$

possibilities.

(40) **Cartesian quadrupole moment of an ellipsoid.**

(a) We have

$$\left(\frac{x^1}{c^1}\right)^2 + \left(\frac{x^2}{c^2}\right)^2 + \left(\frac{x^3}{c^3}\right)^2 = 1 \quad\text{with}\quad c^1 = c^2 = a\,,\;\; c_3 = b$$

$$\text{and charge density }\; \rho(\vec{x}) = \begin{cases}\rho = \text{const} & \text{within the ellipsoid,}\\ 0 & \text{otherwise.}\end{cases}$$

The total charge is

$$q = \int_{\sum_{i=1}^{3}\left(\frac{x^i}{c^i}\right)^2 \le 1} d^3x\,\rho \;=\; c^1\,c^2\,c^3\,\rho \int_{\sum_{i=1}^{3}(x'^i)^2 \le 1} d^3x'$$
$$= \frac{4\pi}{3}\,c^1\,c^2\,c^3\,\rho \;=\; \frac{4\pi}{3}\,a^2\,b\,\rho$$

We write the quadrupole moments of the ellipsoid as

$$Q^{ij} = Q^{ij}_S - Q^{ij}_D \;\;\text{with}\;\; Q^{ij}_S = 3\,\rho \int_{\sum_{i=1}^{3}\left(\frac{x^i}{c^i}\right)^2 \le 1} d^3x\,x^i\,x^j \quad Q^{ij}_D = \rho \int_{\sum_{i=1}^{3}\left(\frac{x^i}{c^i}\right)^2 \le 1} d^3x\,r^2\,\delta^{ij}\,.$$

These moments transform into

$$Q^{ij}_S = 3\,c^1\,c^2\,c^3\,\rho \int_{\sum_{i=1}^{3}(x'^i)^2 \le 1} d^3x'\,c^i\,c^j\,x'^i\,x'^j$$
$$Q^{ij}_D = c^1\,c^2\,c^3\,\rho \int_{\sum_{i=1}^{3}(x'^i)^2 \le 1} d^3x'\delta^{ij} \sum_{k=1}^{3}(c^k\,x'^k)^2\;.$$

We have $Q^{ij}_D = 0$ for $i \ne j$ and the relevant integrals for Q^{ij}_S are of the form

$$\int_{\sum_{i=1}^{3}(x^i)^2 \le 1} d^3x\,x^i\,x^j = \delta^{ij} \int_{\sum_{i=1}^{3}(x^i)^2 \le 1} d^3x\,(x^i)^2 \;\Rightarrow\; Q^{ij} = 0 \;\text{ for }\; i \ne j$$

and

$$\int_{\sum_{i=1}^{3}(x^i)^2 \le 1} d^3x\,(x^i)^2 = \int_{r=|\vec{x}|\le 1} d^3x\,(x^3)^2 = 2\pi \int_0^1 r^4\,dr \int_{-1}^{+1} (\cos\theta)^2\,d\cos\theta = \frac{2\pi}{5}\,\frac{2}{3} = \frac{4\pi}{15}$$

$$\Rightarrow \quad Q^{ii}_S = \frac{q}{5}\,3\,(c^i)^2\,, \quad Q^{ii}_D = \frac{q}{5}\,\sum_{k=1}^{3}(c^k)^2\,.$$

With $c^1 = c^2 = a$ and $c_3 = b$:

$$Q^{11}_S = Q^{22}_S = \frac{3\,q}{5}\,a^2\,, \quad Q^{33}_S = \frac{3\,q}{5}\,b^2\,, \quad Q^{11}_D = Q^{22}_D = Q^{33}_D = \frac{q}{5}\,(2\,b^2 + a^2)$$

resulting into the traceless tensor given by

$$Q^{11} = Q^{22} = \frac{q}{5}\,\left(a^2 - b^2\right), \qquad Q^{33} = \frac{2\,q}{5}\,\left(b^2 - a^2\right)\,.$$

(b) The energy is

$$W = q\,\Phi(0) - \vec{p}\cdot\vec{E}(0) - \frac{1}{6}\sum_{i=1}^{3}\sum_{j=1}^{3} Q_S^{ij}\,\frac{\partial E^j}{\partial x^i}\ .$$

With the normalization $\Phi(0) = 0$ and

$$\vec{p} = \rho \int_{\sum_{i=1}^{3}\left(\frac{x^i}{c^i}\right)^2 \le 1} d^3x\,\vec{x} = 0$$

the result for an external electric field $\vec{E} = E(z)\,\hat{z}$ becomes

$$W = -\frac{1}{6}\,Q_S^{33}\,\frac{d\,E}{d\,z} = -\frac{q\,b^2}{10}\,\frac{d\,E}{d\,z}\ .$$

(41) Exercise E.58: Dielectric sphere.

(a) We expand the potential into spherical harmonics. Due to the axial symmetry we have only $m = 0$ contributions, which are Legendre polynomials.

$$\text{Inside}: \quad \Phi_{\text{in}} = \sum_{l=0}^{\infty} A_l\,r^l\,P_l(\cos\theta)\,,$$

$$\text{Outside}: \quad \Phi_{\text{out}} = \sum_{l=0}^{\infty}\Big[\,B_l\,r^l + C_l\,r^{-(l+1)}\,\Big]\,P_l(\cos\theta)\,.$$

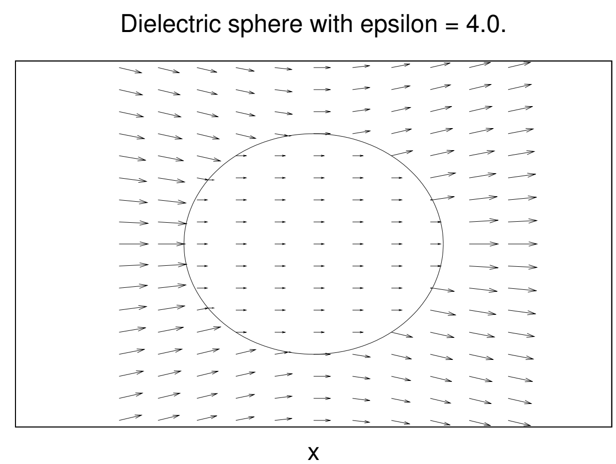
For $r \to \infty$: $\Phi \to -E_0\,z = -E_0\,r\,\cos\theta$. This implies that the only non-vanishing coefficient B_l is

$$B_1 \;=\; -E_0\ .$$

BCs at $r = R$:

$$\text{Tangential}: \quad \frac{1}{R}\,\frac{\partial\Phi_{\text{in}}}{\partial\theta}\bigg|_{r=R} = \frac{1}{R}\,\frac{\partial\Phi_{\text{out}}}{\partial\theta}\bigg|_{r=R}\,,$$

$$\text{Normal}: \quad \epsilon_1\,\frac{\partial\Phi_{\text{in}}}{\partial r}\bigg|_{r=R} = \epsilon_2\,\frac{\partial\Phi_{\text{out}}}{\partial r}\bigg|_{r=R}\,.$$



Matching for the tangential BC $\partial P_l/\partial\theta$ term by term (they are independent functions of θ), we find:

$$A_1\,R \;=\; B_1\,R + \frac{C_1}{R^2} \;\Rightarrow\; A_1 = -E_0 + \frac{C_1}{R^3}$$
$$\text{and for } \; l \geq 2: \quad A_l = \frac{C_l}{R^{(2l+1)}} \;.$$

Similarly we match for the normal BC P_l term by term and find:

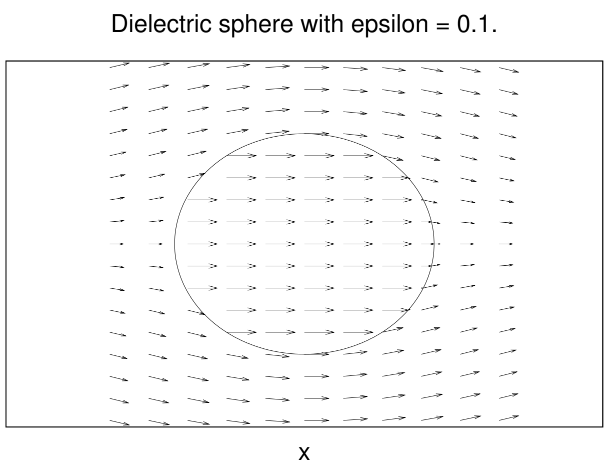
$$\epsilon_1\,A_1 = -\epsilon_2\,E_0 - 2\,\epsilon_2\,\frac{C_1}{R^3}$$
$$\text{and for } \; l \geq 2: \quad \epsilon_1\,l\,A_l = -(l+1)\,\epsilon_2\,\frac{C_l}{R^{(2l+1)}} \;.$$

Putting the $l \geq 2$ equations together we get

$$\frac{C_l}{R^{(2l+1)}} = -\frac{(l+1)\,\epsilon_2}{l\,\epsilon_1}\,\frac{C_l}{R^{(2l+1)}} \;\Rightarrow\; C_l = 0, \quad l \geq 2 \;\Rightarrow\; A_l = 0, \quad l \geq 2 \;.$$

Let us define

$$\epsilon \;=\; \frac{\epsilon_1}{\epsilon_2} \;.$$



With this notation we have

$$\epsilon\, A_1 \;=\; -E_0 - 2\,\frac{C_1}{R^3}\ .$$

Combining this with our other equation for A_1 gives

$$A_1 \;=\; -\left(\frac{3}{\epsilon+2}\right)\,E_0\,, \qquad C_1 \;=\; \left(\frac{\epsilon-1}{\epsilon+2}\right)\,R^3\,E_0\,.$$

Therefore,

$$\Phi_{\rm in} = -\left(\frac{3}{\epsilon+2}\right)\,E_0\,r\,\cos(\theta) \;=\; -\left(\frac{3}{\epsilon+2}\right)\,E_0\,z\,,$$
$$\Phi_{\rm out} = -E_0\,z + \left(\frac{\epsilon-1}{\epsilon+2}\right)\,E_0\,\frac{R^3}{r^2}\,\cos(\theta)\,.$$

Note that the last term is the potential of a dipole. The electric fields are then

$$\vec{E}_{\rm in} = -\nabla\,\Phi_{\rm in} \; = \; \left(\frac{3}{\epsilon + 2}\right)\,\vec{E}_0\,,$$

$$\vec{E}_{\rm out} = -\nabla\,\Phi_{\rm out} \; = \; \vec{E}_0 - \left(\frac{\epsilon - 1}{\epsilon + 2}\right)\,R^3\,\left(\frac{r^2\,\vec{E}_0 - 3\vec{r}\,(\vec{r}\cdot\vec{E}_0)}{r^5}\right)\,.$$

(b) Examples are given in the two figures.
(c) The surface charge density is

$$\begin{aligned}\sigma_{\rm pol} &= \frac{1}{4\pi}\,(E^r_{\rm out} - E^r_{\rm in}) \; = \; \frac{1}{4\pi}\,\hat{r}\cdot\left(\vec{E}_{\rm out} - \vec{E}_{\rm in}\right)\\ &= \frac{1}{4\pi}\,E_0\,\cos(\theta)\,\left[\left(1 - \frac{3}{\epsilon+2}\right) - \left(\frac{\epsilon-1}{\epsilon+2}\right)\,(1-3)\right]\\ &= \frac{3}{4\pi}\,\left(\frac{\epsilon-1}{\epsilon+2}\right)\,E_0\,\cos(\theta)\;.\end{aligned}$$