Electrodynamics A (PHY 5346) Fall 2016 Solutions

Set 10:

(31) Exercise E.39: Paul Trap.

In the quasi-static approximation the field is the electrostatic field with the given boundary conditions at the time in question. Due to the cylindrical symmetry we have $\Phi = \Phi(\rho, z; t)$. To get the boundary conditions on the end electrodes right, we set

$$\Phi \;=\; \left(z^2 - \frac{1}{2}\,\rho^2 - d^2\right)\, F(\rho, z, t)\,.$$

The boundary condition on the ring electrode implies $\rho^2/2 = z^2 + d^2/2$ and on this boundary

$$\Phi \;=\; V_0\,\sin(\omega t) \;=\; -\frac{3}{2}\,d^2\,F(\rho, z, t)\,.$$

Therefore, we try

$$F(t) \;=\; -\frac{2}{3}\,\frac{V_0}{d^2}\,\sin(\omega t)\,,$$

as solution. For it the potential becomes

$$\Phi \;=\; -\frac{2}{3}\,\frac{V_0}{d^2}\,\left(z^2 - \frac{1}{2}\,\rho^2 - d^2\right)\,\sin(\omega t)\,,$$

which is now easily seen to be the solution:

$$\nabla^2\Phi = -\frac{2}{3}\frac{V_0}{d^2}\,\sin(\omega t)\,\nabla^2\left(z^2 - \frac{1}{2}\,\rho^2 - d^2\right) \;=\; 0\,,$$

which is unique due to the Dirichlet BCs. The electric field is given by

$$\vec{E} \;= -\nabla\Phi \;=\; -\left(\hat{\rho}\,\frac{\partial\Phi}{\partial\rho} + \hat{z}\,\frac{\partial\Phi}{\partial z}\right) \;=\; \frac{2}{3}\,\frac{V_0}{d^2}\,\left(2\,z\,\hat{z} - \rho\,\hat{\rho}\right)\,\sin(\omega t)\,.$$

(32) Exercise: Potential of two point charges.

See: Method of images for a sphere, p.50 to 52 lecture notes.

(33) Exercise E.49: Potential inside a sphere with Dirichlet BC.

Given in the problem we have (substitute A_{lm})

$$\Phi = \sum_{l=0}^{\infty} \left(\frac{r}{R}\right)^l \int d\Omega' \sum_{m=-l}^{l} \Phi(\theta', \phi', R)\, \overline{Y}_l^m(\theta', \phi')\, Y_l^m(\theta, \phi)\ .$$

Using the addition formula for spherical harmonics this becomes

$$\Phi = \frac{1}{4\pi} \sum_{l=0}^{\infty} (2l+1)\, \left(\frac{r}{R}\right)^l \int d\Omega'\, P_l(\cos\gamma)\ .$$

Starting with the generating function

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{l=0}^{\infty} P_l(x) t^l$$

we take the derivative on both sides and multiply by $2t$ to get

$$\frac{2xt - 2t^2}{(1 - 2xt + t^2)^{3/2}} = \sum_{l=0}^{\infty} 2l P_l(x) t^l\ .$$

Adding the last two equations gives

$$\begin{aligned} &\frac{2xt - 2t^2}{(1 - 2xt + t^2)^{3/2}} + \frac{1}{\sqrt{1 - 2xt + t^2}} \\ &= \frac{1 - t^2}{(1 - 2xt + t^2)^{3/2}}\ = \sum_{l=0}^{\infty} (2l+1) P_l(x)\, t^l\ . \end{aligned}$$

We define $t = r/R$ and $x = \cos\gamma$ to get

$$\frac{1 - (r/R)^2}{(1 - 2r\cos\gamma - (r/R)^2)^{3/2}} = \sum_{l=0}^{\infty} (2l+1)\, P_l(\cos\gamma)\, \left(\frac{r}{R}\right)^l\ .$$

Finally we substitute back into our second equation to arrive at

$$\Phi = \frac{R\,(R^2 - r^2)}{4\pi} \int d\Omega'\, \frac{\Phi(\theta', \phi', R)}{(R^2 - 2Rr\cos\gamma + r^2)^{3/2}}\ .$$