Electrodynamics A (PHY 5346) Fall 2016 Solutions

Set 4:

(12) Exercise: End of Spacetrip.

We use energy-momentum conservation

$$0 = dp = \begin{pmatrix} dp_r^0 \\ dp_r^1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} dp_e^0 \\ dp_e^1 \\ 0 \\ 0 \end{pmatrix},$$

where the subscript r stands for rocket and e for exhaust. In the temporary rest frame of the rocket we have

$$dp_e^1 = -v\,dm\,, \qquad dp_r^1 = m\,du = m\,g\,d\tau\,,$$

where v is the velocity of the exhaust, dm the infinitesimal change of rest mass of the rocket and du the infinitesimal velocity of the rocket. Note that the exhaust may have no rest mass and our equation for dp_e^1 (no γ) is not obvious. It follows from

$$dp_e^0 = -dp_r^0 = c\,dm \quad \text{and} \quad -\beta = \frac{dp_e^1}{dp_e^0}\,.$$

With this definition $\beta = v/c$ is positive, because dp_e^1 is negative when we choose dp_r^1 positive. Now, separation of variables gives

$$\frac{dm}{m} = -\frac{g}{v}\,d\tau \;\Rightarrow\; \int_{m_0}^{m(\tau)} \frac{dm}{m} = -\frac{g}{v}\int_0^{\tau} d\tau' \;\Rightarrow\; \ln\left(\frac{m(\tau)}{m_0}\right) = -\frac{g\,\tau}{v}\,.$$

(1) The mass of the spaceship decreases with its proper time according to

$$m(\tau) = m_0\,\exp\left(-\frac{g\tau}{v}\right)\,.$$

(2) With $v = 0.66\,c$ and after a trip of twenty years the remaining fraction of the mass is $m(20\,\text{years})/m_0 = 2.68 \times 10^{-14}$.
(3) With $v = c$ it becomes $m(20\,\text{years})/m_0 = 1.1 \times 10^{-9}$.

(13) Exercise: Redshift.

The equation for the redshift is

$$\lambda' \;=\; \lambda\sqrt{\frac{1+\beta}{1-\beta}} \;\Rightarrow\; \left(\frac{\lambda'}{\lambda}\right)^2 \;=\; \frac{1+\beta}{1-\beta}$$

$$\left(\frac{\lambda'}{\lambda}\right)^2 - \beta\left(\frac{\lambda'}{\lambda}\right)^2 \;=\; 1+\beta$$

$$\left(\frac{\lambda'}{\lambda}\right)^2 - 1 \;=\; \beta\left[1+\left(\frac{\lambda'}{\lambda}\right)^2\right] \;\Rightarrow\; \beta \;=\; \frac{(\lambda'/\lambda)^2-1}{(\lambda'/\lambda)^2+1}\,.$$

With $\lambda' = (729.2\,[nm])\,m^2/(m^2-4)$ and $\lambda = (364.56\,[nm])\,m^2/(m^2-4)$ we find

$$\beta \;=\; \frac{v}{c} \;=\; 0.6\,.$$

(14) Exercise: Relativistic energy-momentum conservation.

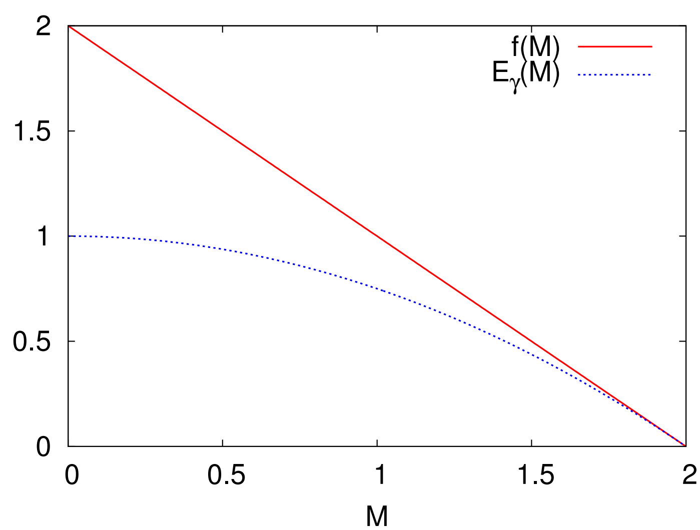

Fig. 0.1 Comparison of $f(M) = 2m - M$ with photon energy $E_\gamma(M)$.

1. The 4-vector energy-momentum conservation reads

$$\begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{M^2+p^2} \\ -p \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} E_\gamma \\ p_\gamma \\ 0 \\ 0 \end{pmatrix}$$

Therefore (eliminating p via $p = p_\gamma = E_\gamma$),

$$2m - E_\gamma = \sqrt{M^2 + E_\gamma^2},$$

$$(2m - E_\gamma)^2 = 4m^2 - 4m\,E_\gamma + E_\gamma^2 = M^2 + E_\gamma^2\,,$$

$$E_\gamma = \frac{4m^2 - M^2}{4m}\,.$$

2. The requested $E_\gamma(M)$ values are: $E_\gamma(0) = m$,

$$E_\gamma\left(\frac{m}{2}\right) = \frac{15\,m}{16}\,,\quad E_\gamma(m) = \frac{3\,m}{4}\,,\quad E_\gamma(\sqrt{2}\,m) = \frac{m}{2}\,,\quad E_\gamma(\sqrt{3}\,m) = \frac{m}{4}\,.$$

3. The sketch is given in the figure.

(15) **Electromagnetic field tensor in $\vec{E}$ and $\vec{B}$ fields.**

A. From the J^0 component we get

$$\partial_0 F^{00} + \partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} = \frac{4\pi}{c}\,J^0 = 4\pi\,\rho = \nabla\cdot\vec{E} = \partial_1 E^1 + \partial_2 E^2 + \partial_3 E^3\,.$$

Therefore, $F^{00} = 0$ by anti-symmetry, $F^{10} = E^1$, $F^{20} = E^2$ and $F^{30} = E^3$.

From the J^1 component we get

$$\partial_0 F^{01} + \partial_1 F^{11} + \partial_2 F^{21} + \partial_3 F^{31} = \frac{4\pi}{c}\,J^1 = \partial_2 B^3 - \partial_3 B^2 - \partial_0 E^1\,.$$

Therefore, $F^{01} = -E^1$ consistent with $F^{10} = E^1$, $F^{11} = 0$ by anti-symmetry, $F^{21} = B^3$ and $F^{31} = -B^2$.

From the J^2 component we get

$$\partial_0 F^{02} + \partial_1 F^{12} + \partial_2 F^{22} + \partial_3 F^{32} = \frac{4\pi}{c}\,J^2 = \partial_3 B^1 - \partial_1 B^3 - \partial_0 E^2\,.$$

Therefore, $F^{02} = -E^2$, $F^{12} = -B^3$ both consistent with anti-symmetry, $F^{22} = 0$ by anti-symmetry and (new) $F^{32} = B^1$.

Using anti-symmetry all components are now determined and the $F^{\alpha\beta}$ field tensor reads

$$\left(F^{\alpha\beta}\right) = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} .$$

The remaining equations from $\beta = 3$ can be used for consistency checks.

B. For $\beta = 0$:

$$\partial_\alpha\, {}^*F^{\alpha 0} = \partial_1\, B^1 + \partial_2\, B^2 + \partial_2\, B^2 = \nabla \cdot \vec{B} = 0\,.$$

For $\beta = 1$:

$$\partial_\alpha\, {}^*F^{\alpha 1} = -\partial_0\, B^1 - \partial_2\, E^3 + \partial_3\, E^2 = -\partial_0\, B^1 - (\nabla \times \vec{E}\,)^1 = 0\,.$$

For $\beta = 2$:

$$\partial_\alpha\, {}^*F^{\alpha 2} = -\partial_0\, B^2 + \partial_1\, E^3 - \partial_3\, E^1 = -\partial_0\, B^2 - (\nabla \times \vec{E}\,)^2 = 0\,.$$

For $\beta = 3$:

$$\partial_\alpha\, {}^*F^{\alpha 3} = -\partial_0\, B^3 - \partial_1\, E^2 + \partial_2\, E^1 = -\partial_0\, B^3 - (\nabla \times \vec{E}\,)^3 = 0\,.$$

So,

$$\nabla \cdot \vec{B} = 0 \quad \text{and} \quad -\partial_0 \vec{B} - \nabla \times \vec{E} = 0\,.$$