Electrodynamics A (PHY 5346) Fall 2016 Solutions

Set 5:

(16) Exercise: Proper time under rotation.

$$c^2\,d\tau^2 = c^2\,dt^2 - \omega^2 R^2\,dt^2 \Rightarrow \frac{d\tau}{dt} = \sqrt{1-\frac{\omega^2 R^2}{c^2}}\ .$$

Now $g = v^2/R = \omega^2\,R$, where $g = 9.81\,[km/s^2]$ and v is the velocity of the satellite. For $R = (6,378 + 160)\,[km]$ (equatorial radius plus $160\,[km]$) we get $\omega = 0.0012249\,[s^{-1}]$, which correpsonds to a period of $T = 2\pi/\omega = 85.5\,[m]$. For the time dilation we find

$$1-\frac{d\tau}{dt} = \frac{1}{2}\left(\frac{\omega\,R}{c}\right)^2 = 0.356321\times 10^{-9}\,.$$

Compared to this the time dilation of a clock on the equator is with

$$1-\frac{d\tau}{dt} = \frac{1}{2}\left(\frac{\omega\,R}{c}\right)^2 = 0.119517\times 10^{-11}$$

negligle. However, gravity effects need to be taken into acount for the final result.

(17) Exercise: Homogeneous Maxwell equations

We choose $\beta = 0$ in $\partial_\alpha\,{}^*F^{\alpha\beta} = 0$ and obtain: $\partial_\alpha\,{}^*F^{\alpha 0} = 0$. Using the definition of the dual tensor, this reads

$$\frac{1}{2}\,\left(\partial_\alpha\,\epsilon^{\alpha 0\gamma\delta}\,F_{\gamma\delta}\right) = 0$$

or

$$\partial_\alpha\,F_{\gamma\delta} + \partial_\gamma\,F_{\delta\alpha} + \partial_\delta\,F_{\alpha\gamma} \;=\; 0 \qquad (0.1)$$

for all three indices different and not 0. Repeating this for $\beta = 1$, 2 and 3, we find that equation (0.1) hold for all combinations of distinct indices α, β and γ. Due to the antisymmetry of electromagnetic tensor, the equation still holds when some of the indices agree. Finally, all indices can of course be raised. This proves the equivalence of the two versions of the homogeneous Maxwell equations. More by brute force, one could just write out all 64 equations.

More elegantly: Use

$$\frac{1}{2}\,\partial_\alpha\,\epsilon_{\hat\alpha\beta\hat\gamma\hat\delta}\,\epsilon^{\alpha\beta\gamma\delta}\,F_{\gamma\delta}=0$$

and expand the contraction in β of the two Levi-Civita symbols into Kronecker deltas.

(18) Exercise: Lorentz transformation of the electric and magnetic fields.

In matrix notation we have $F' = A\,F\,\tilde{A}$, where F is the electromagnetic field tensor, A the Lorentz matrix and $\tilde{A}$ its transpose. We perform the calculation in two steps. First,

$$A\,F \;=\; \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\gamma\beta\,E^1 & -\,\gamma\,E^1 & -\,\gamma\,E^2+\gamma\beta\,B^3 & -\,\gamma\,E^3-\gamma\beta\,B^2 \\ \gamma\,E^1 & \gamma\beta\,E^1 & \gamma\beta\,E^2-\gamma\,B^3 & \gamma\,E^2+\gamma\beta\,B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}$$

then (note $\tilde{A}=A$)

$$F' \;=\; (A\,F)\,\tilde{A} \;=$$

$$\begin{pmatrix} 0 & -E^1 & -\gamma\,(E^2-\beta\,B^3) & -\,\gamma\,(E^3+\beta\,B^2) \\ E^1 & 0 & -\gamma\,(B^3-\beta\,E^2) & \gamma\,(B^2+\beta\,E^3) \\ \gamma\,(E^2-\beta\,B^3) & \gamma\,(B^3-\beta\,E^2) & 0 & -B^1 \\ \gamma\,(E^3+\beta\,B^2) & -\,\gamma\,(B^2+\beta\,E^3) & B^1 & 0 \end{pmatrix}$$

Comparing this with the elements of F' in its standard form, we read off:

$$\begin{aligned} E'^1 &= E^1\,, & B'^1 &= B^1\,, \\ E'^2 &= \gamma\,(E^2-\beta\,B^3)\,, & B'^2 &= \gamma\,(B^2+\beta\,E^3)\,, \\ E'^3 &= \gamma\,(E^3+\beta\,B^2)\,, & B'^3 &= \gamma\,(B^3-\beta\,E^2)\,. \end{aligned}$$

If $\vec{\beta}$ is no longer in $\hat{e}^1$ direction we decompose the fields in components parallel and perpendicular to $\vec{\beta}$ and these equation become

$$\begin{aligned} E'^{\,\parallel} &= E^{\parallel}\,, & B'^{\,\parallel} &= B^{\parallel}\,,\\ \vec{E}'^{\,\perp} &= \gamma\left(\vec{E}^{\perp} + \vec{\beta}\times\vec{B}^{\perp}\right)\,, & \vec{B}'^{\,\perp} &= \gamma\left(\vec{B}^{\perp} - \vec{\beta}\times\vec{E}^{\perp}\right)\,. \end{aligned}$$

To transform this we first note that the equations

$$\vec{\beta}\times\vec{E}^{\perp} = \vec{\beta}\times\vec{E} \quad \text{and} \quad \vec{\beta}\times\vec{B}^{\perp} = \vec{\beta}\times\vec{B}$$

hold, so that

$$\vec{E}'^{\,\perp} = \gamma\left(\vec{E}^{\perp} + \vec{\beta}\times\vec{B}\right) \quad \text{and} \quad \vec{B}'^{\,\perp} = \gamma\left(\vec{B}^{\perp} - \vec{\beta}\times\vec{E}\right)$$

are correct. To extend them to the parallel components, we have to subtract a term that vanishes for the perpendicular components, i.e., it should be proportional to $\vec{\beta}\left(\vec{\beta}\cdot\vec{E}\right)$ in the first and to $\vec{\beta}\left(\vec{\beta}\cdot\vec{B}\right)$ in the second equation. The proportionality factor $\gamma^2/(\gamma+1)$ follows then from the observation

$$\gamma - \frac{\gamma^2\,\beta^2}{\gamma+1} = \frac{\gamma^2+\gamma-\gamma^2\,\beta^2}{\gamma+1} = \frac{\cosh^2\zeta + \cosh\zeta - \sinh^2\zeta}{\cosh\zeta+1} = 1\,.$$