Electrodynamics A (PHY 5346) Fall 2016 Solutions

Set 8:

(24) Exercise (E.40): Method of images for a point charge in front of an infinite grounded plane.

For the grounded plane we may choose the $(x-y)$ plane and position the charge q at $z=+a$ on the z-axis. By symmetry reasons the image charge, which ensures $\Phi=0$ on the plane, is $-q$ at $z=-a$.

(a) By Gauss law $Q_{\rm surface}=-q\,.$

(b) $F=|\vec{F}|=q^2/(2a)^2$ by Coulomb's law.

(c) Using cylindrical coordinates $[x=\rho\cos(\phi),\ y=\rho\sin(\phi)]$, the potential due to charge and image charge is

$$\Phi(\rho,z)=\frac{q}{\sqrt{\rho^2+(z-a)^2}}-\frac{q}{\sqrt{\rho^2+(z+a)^2}}\ .$$

The induced surface charge density is

$$\sigma\ =\ -\frac{1}{4\pi}\left.\frac{\partial\Phi}{\partial z}\right|_{z\to 0^+}=-\frac{q}{4\pi}\lim_{z\to 0^+}\left\{\frac{z+a}{[\rho^2+(z+a)^2]^{3/2}}-\frac{z-a}{[\rho^2+(z-a)^2]^{3/2}}\right\}$$
$$=-\frac{q}{2\pi}\,\frac{a}{[\rho^2+a^2]^{3/2}}\ =\ -\frac{q}{2\pi}\,\frac{a}{r^3}$$

where r is the distance of the point on the plane from the position of the charge q.

(d) The corresponding Dirichlet Green function is

$$G_D(\vec{r},\vec{r}^{\,'})=-\frac{1}{4\pi\sqrt{(\vec{\rho}-\vec{\rho}^{\,\prime})^2+(z-z')^2}}+\frac{1}{4\pi\sqrt{(\vec{\rho}-\vec{\rho}^{\,\prime})^2+(z+z')^2}}\ .$$

with the derivative

$$\left.\frac{\partial G_D}{\partial z'}\right|_{z'=0}\ =\ -\frac{z}{2\pi\ \left[(\vec{\rho}-\vec{\rho}^{\,\prime})^2+z^2\right]^{3/2}}\ .$$

(25) Exercise (E.42): Potential inside a sphere from distinct BCs on the half-spheres.

A. We have to calculate

$$\Phi(r)=R^2\int_S d\Omega'\,\Phi_0\,\left.\frac{\partial G_D}{\partial r'}\right|_{r'=R}\ ,$$

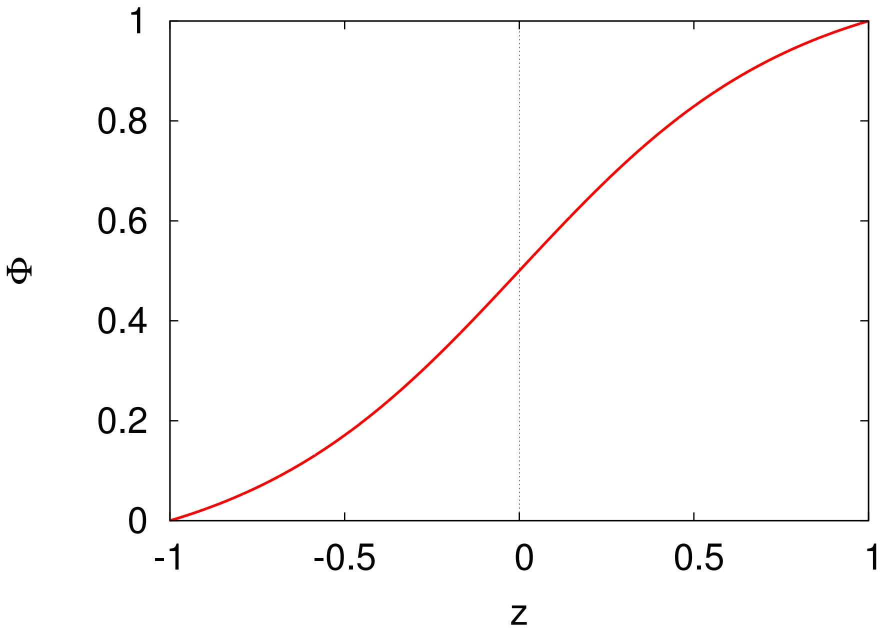

Fig. 0.1 Potential asfunction of z.

where the surface is the upper half-sphere and the derivative of the Green function is given by

$$\left.\frac{\partial G_D}{\partial r'}\right|_{r'=R} = \frac{R^2 - r^2}{4\pi\, R\, \left(R^2 + r^2 - 2Rr\, \cos\gamma\right)^{3/2}}$$

with $\cos\gamma = \cos(\theta)\, \cos(\theta') + \sin(\theta)\, \sin(\theta')\, \cos(\phi - \phi')$. For $r = z$ on the z-axis we have $\theta = 0$ and, hence, $\cos\gamma = \cos\theta'$. The integral over the upper half-sphere becomes

$$\Phi(z) = \Phi_0\, \frac{R}{2}\, (R+z)\, (R-z) \int_0^1 \frac{d\cos\theta'}{\left(R^2 + z^2 - 2R\, z\, \cos\theta'\right)^{3/2}}\,.$$

With $a = R^2 + z^2$ and $b = 2Rz$ the integral is

$$\int_0^1 \frac{dx}{(a - b\,x)^{3/2}} = \left.\frac{2}{b\,(a - b\,x)^{1/2}}\right|_0^1 = \frac{2}{b\sqrt{a-b}} - \frac{2}{b\sqrt{a}}$$
$$= \frac{1}{Rz\sqrt{(R-z)^2}} - \frac{1}{Rz\sqrt{R^2+z^2}}$$

and

$$\Phi(z) = \Phi_0\,\frac{1}{2z}\,(R+z)\left(1 - \frac{R-z}{\sqrt{R^2+z^2}}\right).$$

Special values:

$$\begin{aligned} z = R \;&\Rightarrow\; \Phi(R) = \Phi_0\,\frac{2R}{2R} = \Phi_0\,,\\ z = -R \;&\Rightarrow\; \Phi(-R) = 0\,,\\ z \to 0 \;&\Rightarrow\; \Phi(0) = \Phi_0\,\frac{R}{2z}\left(1 - \frac{R-z}{R}\right) \to \frac{1}{2}\,\Phi_0\,. \end{aligned}$$

B. The plot is shown in the figure.

(26) Exercise (E.46): Potential in a rectangular box.

By separation of variables, we get

$$\Phi(x,y,z) = \sum_{nm} A_{nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sinh\left(z\pi\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}\right)$$

The BC condition given in the problem is

$$\Phi(x,y,c) = \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi y}{b}\right) + \sin\left(\frac{3\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)\;.$$

From this we find that only $(n = 1, m = 2)$ and $(n = 3, m = 1)$ contribute to the expansion

$$\begin{aligned}\Phi(x,y,z) = A_{12}\,&\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi y}{b}\right)\sinh\left(z\pi\sqrt{\frac{1}{a^2}+\frac{4}{b^2}}\right)\\ &+ A_{31}\,\sin\left(\frac{3\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)\sinh\left(z\pi\sqrt{\frac{9}{a^2}+\frac{1}{b^2}}\right)\;.\end{aligned}$$

Comparing to

$$\Phi(x,y,c) = \sin\left(\frac{\pi x}{a}\right)\,\sin\left(\frac{2\pi y}{b}\right) + \sin\left(\frac{3\pi x}{a}\right)\,\sin\left(\frac{\pi y}{b}\right)$$

we find

$$A_{31} = \frac{1}{\sinh\left(c\pi\sqrt{9/a^2+1/b^2}\right)} \quad \text{and} \quad A_{12} = \frac{1}{\sinh\left(c\pi\sqrt{1/a^2+4/b^2}\right)}\ .$$

The same result is obtained by using the integral definitions of the coefficients and performing the integrations.