Electrodynamics A (PHY 5346) Fall 2016 Solutions

Set 11:

(34) Exercise E.48: Symmetry relation for spherical harmonics.

The proof will be by induction from m to $m+1$. The differential operators $L_\pm$ are explicitly given by

$$L_+ = e^{+i\phi}\left(+\frac{\partial}{\partial\theta} + i\,\cot\theta\,\frac{\partial}{\partial\phi}\right),$$
$$L_- = e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + i\,\cot\theta\,\frac{\partial}{\partial\phi}\right).$$

Up to a real factor $Y_l^{\ 0}$ is given by

$$(L_+)^l\,Y_l^{\ -l} = (-1)^l\,(L_+)^l\,\sin^l(\theta)\,e^{-il\phi}\,.$$

It follows that $Y_l^{\ 0}$ is real, because the $\partial/\partial\phi$ differentiation produces $-il\exp(-il\phi)$, so that the $i\exp(+i\phi)$ factor in the $\partial/\partial\phi$ part of L_+ multiplies with $il\exp(-il\phi)$ to -1. Therefore, the relation

$$Y_l^{-m} \;=\; (-1)^m\,\overline{Y}_l^{\ m}$$

holds for $m=0$, i.e., $Y_l^{\ 0} = \overline{Y}_l^{\ 0}$. By induction hypothesis we have (after multiplying both sides by L_-)

$$L_- Y_l^{-m} = (-1)^m L_-\overline{Y}_l^{\ m}\,.$$

The explicit form of the differential operators shows $\overline{L}_+ = -L_-$. Hence,

$$L_- Y_l^{-m} = (-1)^{m+1}\overline{L}_+\overline{Y}_l^{\ m} = (-1)^{m+1}\sqrt{(l-m)\,(l+m+1)}\;\overline{Y}_l^{\ m+1}$$

holds together with

$$L_- Y_l^{-m} = \sqrt{(l-m)\,(l+m+1)}\;Y_l^{-m-1}\,.$$

Comparing the right-hand sides of the last two equations gives

$$Y_l^{-m-1} \;=\; (-1)^{m+1}\,\overline{Y}_l^{\ m+1}$$

and, hence, proves the induction.

(35) Exercise E.51: Dirichlet Green function of a cylinder.

(a) We want to solve the eigenvalue problem

$$\frac{\partial^2 \Phi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \Phi}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} \;=\; -\lambda^2\,\Phi$$

with $\Phi = 0$ on the surface of the cylinder. We assume the cylinder to be of radius ρ_0 about the z-axis and of length L from $z = 0$ to $z = L$. The separation ansatz $\Phi = R(\rho)Q(\phi)Z(z)$ gives

$$\frac{d^2 Z}{dz^2} + \alpha^2\,Z \;=\; 0\,, \qquad \frac{d^2 Q}{d\phi^2} + m^2\,Q \;=\; 0 \quad \text{and}$$

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{m^2}{\rho^2}\right) R \;=\; 0$$

with $m = 0, 1, 2, \ldots$ and the constraint

$$\alpha^2 + k^2 \;=\; \lambda^2\;.$$

The BC implies then

$$Z \sim \sin(\alpha_j z),\;\; \alpha_j = \frac{\pi j}{L},\;\; j = 1, 2, \ldots\,, \qquad Q \sim e^{\pm im\phi},\;\; m = 0, 1, 2, \ldots$$

$$\text{and} \quad R \sim J_m(k_{mn}\rho),\;\; k_{mn} = \frac{x_{mn}}{\rho_0}\,,$$

where x_{mn} are the roots of the Bessel function of order m, $J_m(x_{mn}) = 0$, $n = 1, 2, \ldots$. The eigenfunctions with eigenvalues

$$\lambda^2_{jmn} = \alpha_j^2 + k_{mn}^2$$

are

$$\Phi_{jmn} \;=\; A_{jmn}\,\sin(\alpha_j z)\,e^{\pm im\phi}\,J_m(k_{mn}\rho)\,,$$

where the A_{jmn} are normalization constants so that

$$\int_0^{\rho_0} \rho\,d\rho \int_0^{2\pi} d\phi \int_0^L dz\,\overline{\Phi}_{j'm'n'}(\rho,\phi,z)\,\Phi_{jmn}(\rho,\phi,z) \;=\delta_{j'j}\,\delta_{m'm}\,\delta_{n'n}$$

holds. Now for $j' = j$, $m' = m$ and $n' = n$

$$\int_0^{\rho_0} \rho\,d\rho\,J_m\left(\frac{x_{mn}}{\rho_0}\,\rho\right)^2 \;=\; \frac{\rho_0^2}{2}\,J_{m+1}\left(x_{mn}\right)^2\,,$$

$$\int_0^{2\pi} d\phi \, e^{i(m'-m)} \;=\; \int_0^{2\pi} d\phi \;=\; 2\pi \,,$$

$$\int_0^L dz \, \sin^2\left(\frac{j\pi}{L}\, z\right) = \frac{1}{2}\int_0^L dz \, \left[\sin^2\left(\frac{j\pi}{L}\, z\right) + \cos^2\left(\frac{j\pi}{L}\, z\right)\right] = \frac{L}{2}$$

and in the convention $\nabla^2 G_D = \delta(\vec{x}\,' - \vec{x})$ the properly normalized Green function is

$$G_D\,(\vec{r}\,', \vec{r}) \;=\; -\frac{2}{\pi\, L\, \rho_0^2} \sum_{j=1}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{\lambda_{jmn}^{-2}}{J_{m+1}^2(x_{mn})} \times$$

$$\sin(\alpha_j z')\, \sin(\alpha_j z)\, e^{im\,(\phi'-\phi)}\, J_m(k_{mn}\rho')\, J_m(k_{mn}\rho) \;.$$

(36) Exercise E.52: Point charge at the center of a box.

(a) The separation ansatz

$$\Phi_{lmn}(\vec{r}) = F_l(x) G_m(y) H_n(z)$$

gives the solutions of the eigenvalue problem

$$(\nabla^2 + k_{lmn}^2)\, F_l(x)\, G_m(y)\, F_n(z) = 0$$

with BCs $\Phi_{lmn} = 0$:

$$F_k(\xi) \;=\; G_k(\xi) \;=\; H_k(\xi) \;=\; \sqrt{\frac{2}{a}} \begin{cases} \sin(k\pi\xi/a) & \text{for} \quad k = 2, 4, \ldots\,, \\ \cos(k\pi\xi/a) & \text{for} \quad k = 1, 3, \ldots\,. \end{cases}$$

In the convention $\nabla^2 G_D = \delta(\vec{x}\,' - \vec{x})$ the Dirichlet Green function is then given by

$$G_D\,(\vec{r}\,', \vec{r}) \;=\; -\sum_{l,m,n=1}^{\infty} \frac{F_l(x')\, G_m(y')\, H_n(z')\, F_l(x)\, G_m(y)\, H_n(z)}{k_{lmn}^2}$$

$$\text{with} \quad k_{lmn}^2 = \frac{\pi^2}{a^2}\, \left(l^2 + m^2 + n^2\right) \;.$$

(b) For a given charge distribution is the box and given BCs on its surface the potential inside the box is then

$$\Phi(\vec{r}) \;=\; -4\pi \int_V d^3x'\,\rho(\vec{r}')\,G_D\,(\vec{r}',\vec{r}) + \int_S d^3a'\,\Phi\,\frac{\partial}{\partial n'}G_D\ .$$

For our case the BC on the surface is $\Phi = 0$ and the second integral vanishes. The charge distribution is a point charge at the origin, so that

$$\begin{aligned}\Phi(\vec{r}) &= -4\pi \int_V d^3x'\,q\,\delta(\vec{r}')\,G_D\,(\vec{r}',\vec{r}) \\ &= \frac{32\,q}{\pi\,a} \sum_{l,m,n=1,3,\ldots} \frac{\cos(l\pi x/a)\,\cos(m\pi y/a)\,\cos(n\pi z/a)}{l^2+m^2+n^2}\end{aligned}$$

(c) The induced charge density on the $z = a/2$ surface is

$$\begin{aligned}\sigma\left(z=\frac{a}{2}\right) \;&=\; \frac{1}{4\pi}\;\frac{\partial\Phi}{\partial z}\bigg|_{z=a/2} \\ =\; &\frac{32\,q}{4\pi^2\,a} \sum_{l,m,n=1,3,\ldots} \frac{\cos(l\pi x/a)\,\cos(m\pi y/a)\,\sin(n\pi/2)\,(-n\pi/a)}{l^2+m^2+n^2} \\ =\; &\frac{8\,q}{\pi\,a^2} \sum_{l,m,n=1,3,\ldots} n\,(-1)^{\frac{n+1}{2}}\,\frac{\cos(l\pi x/a)\,\cos(m\pi y/a)}{l^2+m^2+n^2}\ .\end{aligned}$$