Electrodynamics A (PHY 5346) Fall 2016 Solutions

Set 9:

(27) Exercise E.41: Potential over a plane.

The Dirichlet Green function is obtained with the method of images

$$G_D(\vec{r},\vec{r}') = -\frac{1}{4\pi\sqrt{(\vec{\rho}-\vec{\rho}')^2+(z-z')^2}} + \frac{1}{4\pi\sqrt{(\vec{\rho}-\vec{\rho}')^2+(z+z')^2}} \ .$$

and

$$\left.\frac{\partial G_D}{\partial z'}\right|_{z'=0} = -\frac{z}{2\pi\ \left[(\vec{\rho}-\vec{\rho}')^2+z^2\right]^{3/2}} \ .$$

The potential on the z axis is then

$$\Phi(\vec{r}) = -\int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} dy'\, \Phi(x',y',0)\ \left.\frac{\partial G_D}{\partial z'}\right|_{z'=0}$$

where the potential $\Phi(x',y',0)$ in the x-y-plane is given. In our case the integration is only over the upper half-plane as $\Phi=0$ on the lower half-plane. On the z-axis $\rho=0$ holds and

$$\Phi(z) = \int_{-\infty}^{+\infty} dx' \int_0^{\infty} dy'\ \frac{z\,\Phi_1}{2\pi\ \left[(\vec{\rho}')^2+z^2\right]^{3/2}}$$
$$= \frac{z\,\Phi_1}{2\pi}\int_0^{\pi} d\phi' \int_0^{\infty} \rho'\, d\rho'\, \frac{1}{(\rho'^2+z^2)^{3/2}} = \frac{\Phi_1}{2} \,.$$

(28) Exercise E.44: Angled plates.

(a) The potential satisfies $\nabla^2\Phi=0$. Choosing cylindrical coordinates and recognizing that by symmetry reasons Φ does not depend on ρ and z, we have

$$\nabla^2\Phi = \frac{1}{\rho^2}\frac{d^2\Phi}{d\phi^2} = 0\,.$$

(b) The boundary conditions are $\Phi(0)=0$ and $\Phi(\beta)=\Phi_0$ and the solution is

$$\Phi(\phi) = a\,\phi + b = \Phi_0\,\frac{\phi}{\beta}\,.$$

(c) The electric field is

$$\vec{E} \;=\; -\nabla\Phi \;=\; -\frac{\hat{\phi}}{\rho}\,\frac{d\Phi}{d\phi} \;=\; -\frac{\Phi_0}{\beta\,\rho}\,\hat{\phi}\,.$$

(d) Charge densities and total charge follow from Gauss' law. On each plate with opposite signs:

$$\int_S d\vec{a}\cdot\vec{E} \;=\; 4\pi\,Q\,, \qquad -\hat{\phi}\cdot\vec{E} \;=\; 4\pi\,\sigma\,.$$

Therefore,

$$\frac{\Phi_0}{\beta\,\rho}\,\hat{\phi} = 4\pi\,\sigma \;\Rightarrow\; \sigma = \frac{\Phi_0}{4\pi\,\beta\,\rho}$$

and the total charge on each plate is

$$Q \;=\; \int_{\rho_1}^{\rho_2} d\rho \int_0^h dz\,\frac{\Phi_0}{4\pi\,\beta\,\rho} \;=\; \frac{\Phi_0\,h}{4\pi\,\beta}\,\ln(\rho_2/\rho_1)\,.$$

(d) The capacitance is

$$C \;=\; \frac{Q}{\Phi_0} \;=\; \frac{h}{4\pi\,\beta}\,\ln(\rho_2/\rho_1)\,.$$

(e) The energy is

$$U \;=\; \frac{1}{2}\,C\,\Phi_0^2 \;=\; \frac{h\,\Phi_0^2}{8\pi\,\beta}\,\ln(\rho_2/\rho_1)\,.$$

From this the torque follows to be

$$\vec{\tau} \;=\; \frac{dU}{d\beta}\,\hat{z} \;=\; -\frac{h\,\Phi_0^2}{8\pi\,\beta^2}\,\ln(\rho_2/\rho_1)\,\hat{z}\,.$$

(29) Exercise: Expansion in cylindrical coordinates.

As there is no azimuthal dependence in the BC, there will be not ϕ dependence in the result and the expansion reads

$$\Phi(\rho,z) \;=\; \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn}\,J_m(k_{mn}\rho)\,\sinh(k_{mn}z)$$

where due to the BC the only surviving coefficient is A_{01}. So,

$$\Phi(\rho,z) \;=\; A_{01}\,J_0(k_{01}\rho)\,\sinh(k_{01}z)\,.$$

Now,

$$1 \; = \; \rho_0 \; = \; \frac{x_{01}}{k_{01}} \; \Rightarrow \; k_{01} \; = \; x_{01} \; \approx 2.405$$

and the BC at $z = 1$ is matched with $A_{01} = 1/\sinh(x_{01})$.

(a) The potential inside the cylinder is

$$\Phi(\rho, z) \; = \; J_0(x_{01}\rho) \, \frac{\sinh(x_{01}z)}{\sinh(x_{01})} \, .$$

(b) For $\rho = 0$, $z = 1/2$: $J_0(x_{01}\rho) = J_0(0) = 1$, $\sinh(x_{01}/2) = 1.514$. Further, $\sinh(x_{01}) = 5.494$ and we find

$$\Phi\left(0, \frac{1}{2}\right) \; = \; J_0(0) \, \frac{1.514}{5.495} \; = \; 0.2756 \, .$$

(c) For $\rho = 1/2$ we have $J_0(x_{01}/2 = 0.670)$ and find

$$\Phi\left(\frac{1}{2}, \frac{1}{2}\right) \; = \; 0.670 \, \frac{1.514}{5.495} \; = \; 0.1846, .$$

(30) Exercise: Calculate L_-. See (C.17) to (C.23) lecture notes.