Electrodynamics A (PHY 5346) Fall 2016 Solutions

Set 3:

(7) Exercise (CW 5 points): Lorentz group Lie matrix generator.

The Matrix L is:

$$L = \begin{pmatrix} l^0{}_0 & l^0{}_1 & l^0{}_2 & l^0{}_3 \\ l^1{}_0 & l^1{}_1 & l^1{}_2 & l^1{}_3 \\ l^2{}_0 & l^2{}_1 & l^2{}_2 & l^2{}_3 \\ l^3{}_0 & l^3{}_1 & l^3{}_2 & l^3{}_3 \end{pmatrix}.$$

(1) The matrix g is:

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

By matrix multiplication:

$$-gL = \begin{pmatrix} -l^0{}_0 & -l^0{}_1 & -l^0{}_2 & -l^0{}_3 \\ l^1{}_0 & l^1{}_1 & l^1{}_2 & l^1{}_3 \\ l^2{}_0 & l^2{}_1 & l^2{}_2 & l^2{}_3 \\ l^3{}_0 & l^3{}_1 & l^3{}_2 & l^3{}_3 \end{pmatrix}.$$

(2) The transpose of L is:

$$\widetilde{L} = \begin{pmatrix} \widetilde{l}_0{}^0 & \widetilde{l}_0{}^1 & \widetilde{l}_0{}^2 & \widetilde{l}_0{}^3 \\ \widetilde{l}_1{}^0 & \widetilde{l}_1{}^1 & \widetilde{l}_1{}^2 & \widetilde{l}_1{}^3 \\ \widetilde{l}_2{}^0 & \widetilde{l}_2{}^1 & \widetilde{l}_2{}^2 & \widetilde{l}_2{}^3 \\ \widetilde{l}_3{}^0 & \widetilde{l}_3{}^1 & \widetilde{l}_3{}^2 & \widetilde{l}_3{}^3 \end{pmatrix} = \begin{pmatrix} l^0{}_0 & l^1{}_0 & l^2{}_0 & l^3{}_0 \\ l^0{}_1 & l^1{}_1 & l^2{}_1 & l^3{}_1 \\ l^0{}_2 & l^1{}_2 & l^2{}_2 & l^3{}_2 \\ l^0{}_3 & l^1{}_3 & l^2{}_3 & l^3{}_3 \end{pmatrix}.$$

(3) By matrix multiplication:

$$\widetilde{L}g = \begin{pmatrix} l^0{}_0 & -l^1{}_0 & -l^2{}_0 & -l^3{}_0 \\ l^0{}_1 & -l^1{}_1 & -l^2{}_1 & -l^3{}_1 \\ l^0{}_2 & -l^1{}_2 & -l^2{}_2 & -l^3{}_2 \\ l^0{}_3 & -l^1{}_3 & -l^2{}_3 & -l^3{}_3 \end{pmatrix}.$$

(4) Set $-gL = \widetilde{L}g$ and compare the components:

$$\begin{pmatrix} -l^0{}_0 = l^0{}_0 & -l^0{}_1 = -l^1{}_0 & -l^0{}_2 = -l^2{}_0 & -l^0{}_3 = -l^3{}_0 \\ l^1{}_0 = l^0{}_1 & l^1{}_1 = -l^1{}_1 & l^1{}_2 = -l^2{}_1 & l^1{}_3 = -l^3{}_1 \\ l^2{}_0 = l^0{}_2 & l^2{}_1 = -l^1{}_2 & l^2{}_2 = -l^2{}_2 & l^2{}_3 = -l^3{}_2 \\ l^3{}_0 = l^0{}_3 & l^3{}_1 = -l^1{}_3 & l^3{}_2 = -l^2{}_3 & l^3{}_3 = -l^3{}_3 \end{pmatrix} .$$

The resulting matrix is:

$$\begin{pmatrix} 0 & l^0{}_1 & l^0{}_2 & l^0{}_3 \\ l^0{}_1 & 0 & l^1{}_2 & l^1{}_3 \\ l^0{}_2 & -l^1{}_2 & 0 & l^2{}_3 \\ l^0{}_3 & -l^1{}_3 & -l^2{}_3 & 0 \end{pmatrix} .$$

(8) Exercise (5 points): Obtain the result of the previous Exercise discussing the elements of $g^{\alpha\beta}\,\tilde{l}_\beta{}^\gamma\, g_{\gamma\delta} = -l^\alpha{}_\delta$. Performing the contractions we get $\tilde{l}^\alpha{}_\delta = -l^\alpha{}_\delta$. Using now the definition $\tilde{l}^\alpha{}_\delta = l_\delta{}^\alpha$, the following equations are obtained (no summations, $a = 0, 1, 2, 3$, $i = 1, 2, 3$ and $j = 1, 2, 3$):

$$\tilde{l}^a{}_a = l_a{}^a = +l^a{}_a \Rightarrow l^a{}_a = -l^a{}_a = 0 \ ,$$

$$\tilde{l}^i{}_0 = l_0{}^i = -l^0{}_i \Rightarrow l^0{}_i = +l^i{}_0 \ ,$$

$$\tilde{l}^i{}_j = l_j{}^i = +l^j{}_i \Rightarrow l^j{}_i = -l^i{}_j$$

and the resulting matrix is the same as before.

(9) Exercise: A rotation and a boost generator.

For S_3 we have

$$S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad (S_3)^2 = -\mathbf{1}_{2,3} \quad \text{with} \quad \mathbf{1}_{2,3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

Therefore, for $n = 1, 2, 3, \ldots$

$$(S_3)^{2n} = (-1)^n\, \mathbf{1}_{2,3} \quad \text{and} \quad (S_3)^{2n+1} = (-1)^n\, S_3$$

and the result follows:

$$e^{-\omega S_3} = \mathbf{1} + \sum_{n=1}^{\infty}(-1)^n \frac{\omega^{2n}}{(2n)!}\,\mathbf{1}_{2,3} - \sum_{n=0}^{\infty}(-1)^n \frac{\omega^{2n+1}}{(2n+1)!}\,S_3$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & \sin\omega & 0 \\ 0 & -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Similarly we deal with K_1:

$$K_1 = \begin{pmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix} \quad\text{and}\quad (K_1)^2 = \mathbf{1}_{1,2} \quad\text{with}\quad \mathbf{1}_{1,2} = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}.$$

Therefore, for $n = 1, 2, 3, \dots$

$$(K_1)^{2n} = \mathbf{1}_{1,2} \quad\text{and}\quad (K_1)^{2n+1} = K_1$$

and

$$e^{-\zeta K_1} = \mathbf{1} + \sum_{n=1}^{\infty} \frac{\omega^{2n}}{(2n)!}\,\mathbf{1}_{1,2} - \sum_{n=0}^{\infty} \frac{\omega^{2n+1}}{(2n+1)!}\,K_1 = \begin{pmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(10) Exercise: Four-dimensional Levi-Civita tensor.

(1) Using the Einstein summation convention of the left-hand side, but not on the right-hand side, of the following equation we have

$$g_{\alpha_1\alpha_2} g_{\beta_1\beta_2} g_{\gamma_1\gamma_2} g_{\delta_1\delta_2} \epsilon^{\alpha_2\beta_2\gamma_2\delta_2} = g_{\alpha_1\alpha_1} g_{\beta_1\beta_1} g_{\gamma_1\gamma_1} g_{\delta_1\delta_1} \epsilon^{\alpha_1\beta_1\gamma_1\delta_1}.$$

The right-hand side is zero unless $\alpha_1\beta_1\gamma_1\delta_1$ are a permutation $\pi_0\pi_1\pi_2\pi_3$ of the numbers $0, 1, 2, 3$ and in that case

$$\epsilon_{\pi_0\pi_1\pi_2\pi_3} = g_{\pi_0\pi_0} g_{\pi_1\pi_1} g_{\pi_1\pi_2} g_{\pi_1\pi_3} \epsilon^{\pi_0\pi_1\pi_2\pi_3} = (-1)^3\, \epsilon^{\pi_0\pi_1\pi_2\pi_3},$$

which proves $\epsilon_{\alpha\beta\gamma\delta} = -\epsilon^{\alpha\beta\gamma\delta}$.

(2) We have $\epsilon_{\alpha\beta\gamma_1\delta_1}\epsilon^{\alpha\beta\gamma_2\delta_2} = 0$ unless either $\gamma_1 = \gamma_2$, $\delta_1 = \delta_2$ or $\gamma_1 = \delta_2$, $\delta_1 = \gamma_2$ or (otherwise one of the already used numbers will be repeated by α or β of the sums). Therefore,

$$\epsilon_{\alpha\beta\gamma_1\delta_1}\epsilon^{\alpha\beta\gamma_2\delta_2} = a\,\delta_{\gamma_1}^{\ \gamma_2}\,\delta_{\delta_1}^{\ \delta_2} + b\,\delta_{\gamma_1}^{\ \delta_2}\,\delta_{\delta_1}^{\ \gamma_2}$$

holds. With no summation in the permutation indices π_0 to π_3 the constants follow from

$$\epsilon_{\alpha\beta\pi_2\pi_3}\epsilon^{\alpha\beta\pi_2\pi_3} = \epsilon_{\pi_0\pi_1\pi_2\pi_3}\epsilon^{\pi_0\pi_1\pi_2\pi_3} + \epsilon_{\pi_1\pi_0\pi_2\pi_3}\epsilon^{\pi_1\pi_0\pi_2\pi_3} = -2 = a\,,$$
$$\epsilon_{\alpha\beta\pi_2\pi_3}\epsilon^{\alpha\beta\pi_3\pi_2} = \epsilon_{\pi_0\pi_1\pi_2\pi_3}\epsilon^{\pi_0\pi_1\pi_3\pi_2} + \epsilon_{\pi_1\pi_0\pi_2\pi_3}\epsilon^{\pi_1\pi_0\pi_3\pi_2} = +2 = b\,.$$

(3) We have $\epsilon_{\alpha\beta_1\gamma_1\delta_1}\epsilon^{\alpha\beta_2\gamma_2\delta_2} = 0$ unless $\beta_2\gamma_2\delta_2$ is a permutation of $\beta_1\gamma_1\delta_1$. There are six such permutations, so that the results is a sum of the form

$$\begin{aligned}\epsilon_{\alpha\beta_1\gamma_1\delta_1}\epsilon^{\alpha\beta_2\gamma_2\delta_2} &= a_1\,\delta_{\beta_1}^{\beta_2}\delta_{\gamma_1}^{\gamma_2}\delta_{\delta_1}^{\delta_2} + a_2\,\delta_{\beta_1}^{\gamma_2}\delta_{\gamma 1}^{\delta_2}\delta_{\delta_1}^{\beta_2} + a_3\,\delta_{\beta_1}^{\delta_2}\delta_{\gamma_1}^{\beta_2}\delta_{\delta_1}^{\gamma_2}\\ &+ b_1\,\delta_{\beta_1}^{\beta_2}\delta_{\gamma_1}^{\delta_2}\delta_{\delta_1}^{\gamma_2} + b_2\,\delta_{\beta_1}^{\delta_2}\delta_{\gamma_1}^{\gamma_2}\delta_{\delta_1}^{\beta_2} + b_3\,\delta_{\beta_1}^{\gamma_2}\delta_{\gamma_1}^{\beta_2}\delta_{\delta_1}^{\delta_2}\,,\end{aligned}$$

where it follows from (no summation) $\epsilon_{\pi_0\pi_1\pi_2\pi_3}\epsilon^{\pi_0\pi_1\pi_2\pi_3} = -1$ that $a_i = -1$ and $b_i = 1$ holds for $i = 1, 2, 3$.

(11) Exercise: Addition theorem for transverse velocity components.

In K' the motion with velocity $\vec{u'}$ is

$$x'^i \;=\; c^{-1}\,u'^i\,x'^0\,.$$

With respect to frame K the origin of K' frame moves with speed v along the x^1 axis of K. For $i = 2, 3$ the Lorentz transformations give then

$$x^i \;=\; x'^i \;=\; c^{-1}\,u'^i\,\gamma\,(x^0 - \beta\,x^1)\,.$$

Dividing by x^0 gives the velocity components in K:

$$c^{-1}\,u^i = c^{-1}\,u'^i\,\gamma\,(1 - \beta\,c^{-1}\,u^1) \;=\; c^{-1}\,u'^i\,\gamma\,\left(1 - \beta\,c^{-1}\,\frac{u'^1 + v}{1 + u'^1 v/c^2}\right)\,,$$

where in the second step the already calculated equation for u^1 has been inserted. Bringing everything to the common denominator gives

$$\begin{aligned}u^i &= u'^i\,\gamma\,\frac{(1 + \beta\,u'^1/c) - \beta\,c^{-1}\,(u'^1 + v)}{1 + u'^1 v/c^2}\\ &= u'^i\,\gamma\,\frac{1 - \beta^2}{1 + u'^1 v/c^2} \;=\; \frac{u'^i}{\gamma\,(1 + u'^1 v/c^2)}\,,\end{aligned}$$

where $1 - \beta^2 = 1/\gamma^2$ was used in the last step.