Electrodynamics A (PHY 5346) Fall 2016 Solutions

Set 7:

(21) Exercise: Electric and magnetic fields for an infinitely long, straight wire.

(a) We consider a cylinder of length L and radius ρ, which is centered around the $\hat{z}$ axis. The charge in the cylinder is $Q = L\,\lambda$ and the surface area of its mantel is $A = 2\pi\,\rho\,L$. By symmetry reasons Gauss's law gives then

$$E(\rho)\,2\,\pi\,\rho\,L = 4\pi\,Q = 4\pi\,L\,\lambda \;\Rightarrow\; E(\rho) = 2\,\frac{\lambda}{\rho}$$

and as vector

$$\vec{E} = E(\rho)\,\hat{\rho} = 2\,\frac{\lambda}{\rho}\,\hat{\rho}\;.$$

(b) Lorentz transformation gives

$$\vec{E}^{\,\prime} = \gamma\,\vec{E} \;\;\text{and}\;\; \vec{B}^{\,\prime} = \gamma\,E(\rho)\,\frac{v}{c}\,\hat{z}\times\hat{\rho} = \gamma\,E(\rho)\,\frac{v}{c}\,\hat{\phi} = \gamma\,\frac{v}{c}\,\frac{2\,\lambda}{\rho}\,\hat{\phi}\;.$$

Introducing the current $I = v\,\lambda$ and taking the non-relativistic limit $\gamma \to 1$, we obtain the familiar result of magnetostatics (a special case of the Biot-Savart law)

$$\vec{B}^{\,\prime} = \frac{2}{c}\,\frac{I}{\rho}\,\hat{\phi}\;.$$

(22) Exercise: Charge density from a potential (hydrogen atom).

Laplace's equation for the problem is

$$\nabla^2\Phi(r) = -4\pi\,\rho(\vec{r}) \quad \text{with} \quad \Phi(r) = \frac{q}{r}\,e^{-\alpha r}\left(1+\frac{\alpha r}{2}\right)\;.$$

For $r \neq 0$ we use the Laplace operator in spherical coordinates and have for the continuous part

$$\nabla^2\Phi(r) = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right)\;.$$

It is straightforward, but a bit lengthy, to calculate the derivatives of the potential (algebraic programs like Maple and Mathematica are well-suited for such tasks). The results are

$$r^2 \frac{d\Phi}{dr} = -q\, e^{-\alpha r} \, \left(1 + \alpha r + \alpha^2 r^2/2\right) \,,$$

$$\frac{1}{r^2}\frac{d}{dr}\,\left(r^2\frac{d\Phi}{dr}\right) = +\frac{\alpha^3 q}{2}\, e^{-\alpha r} \,.$$

The singular part is:

$$q\, e^{-\alpha r}\, \left(1 + \frac{\alpha r}{2}\right) \nabla^2 \frac{1}{r} = q\, e^{-\alpha r}\, \left(1 + \frac{\alpha r}{2}\right) \left[-4\pi\, \delta\,(\vec{r}\,)\right] = -4\pi\, q\, \delta(\vec{r}) \,.$$

Combining the results gives

$$\rho(\vec{r}) = q\, \delta\,(\vec{r}) - \frac{\alpha^3 q}{8\pi}\, e^{-\alpha r} \,.$$

The nucleus has the point charge $+q$, which is surrounded by a continuous charge distribution of charge

$$-q = -\int d^3x\, \frac{\alpha^3 q}{8\pi}\, e^{-\alpha r} \,.$$

The charge density is the one of the H_2 wave function, $\rho \sim |\psi|^2$. This separation into a singular and a smooth part is used for Ewald sums in molecular dynamics simulations. See, for instance, chapter 12 of Frenkel and Smit.

(23) Exercise: Symmetry of Green functions which fulfill Dirichlet BCs.

Choosing d^3y as integration variable, Green's theorem reads

$$\int_V (\phi\, \nabla_y^2 \psi - \psi\, \nabla_y^2 \phi)\, d^3y = \int_S (\phi\, \frac{\partial \psi}{\partial n} - \psi\, \frac{\partial \phi}{\partial n})\, da \;.$$

Let $\phi = G_D(\vec{y}, \vec{x}\,)$ and $\psi = G_D(\vec{y}, \vec{x}\,')$, so that

$$\nabla_y^2 \phi = \delta(\vec{y} - \vec{x}) \,, \qquad \nabla_y^2 \psi = \delta(\vec{y} - \vec{x}\,')$$

holds. We find

$$\int_V \left[G_D(\vec{y}, \vec{x}\,)\, \delta(\vec{y} - \vec{x}\,') - G_D(\vec{y}, \vec{x}\,')\, \delta(\vec{y} - \vec{x})\right]\, d^3y = 0$$

because for $\vec{y} \in S$ (i.e., $\vec{y}$ on the surface) $G_D(\vec{y}, \vec{x}\,) = G_D(\vec{y}, \vec{x}^{\,'}) = 0$. Performing the integration over d^3y gives the desired result

$$G_D(\vec{x}^{\,'}, \vec{x}\,) = G_D(\vec{x}, \vec{x}^{\,'}) \ .$$