Electrodynamics A (PHY 5346) Fall 2016 Solutions Final December 14.

1. Electron-positron annihilation.

   Let us use natural units with $c = 1$ and denote the four-vectors of the photons by p and q. Energy conservation:

   $$p^0 + q^0 = 2E \quad \text{with} \quad E = 0.511\,[MeV]\,.$$

   Momentum conservation and massless photons:

   $$\vec{p} = -\vec{q} \quad \text{and} \quad p^2 = q^2 = 0\,.$$

   (a) Therefore, we have in the frame where the photon with momentum p moves along the positive x^1 direction

   $$(p^\alpha) = \begin{pmatrix} E \\ E \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad (q^\alpha) = \begin{pmatrix} E \\ -E \\ 0 \\ 0 \end{pmatrix}\,.$$

   (b) $\beta = 3/5 \;\rightarrow\; \gamma = 1/\sqrt{1-\beta^2} = 5/4$ and the energy is the zero component of a four-vector transforms according to

   $$E' = \gamma\,(1-\beta)\,E = \frac{1}{2}\,E = 0.2555\,[MeV]\,.$$

2. **Coaxial cable.**

   Ampére's law reads

   $$\nabla \times \vec{B} \;=\; \frac{4\pi}{c}\,\vec{J}$$

   where in our case $\vec{J} = J\,\hat{z}$ and

   $$J \;=\; J(\rho) \;=\; \begin{cases} J_1 = I/(\pi\,\rho_1^2) \;\; \text{for} \;\; \rho \le \rho_1\,, \\ 0 \;\; \text{for} \;\; \rho_1 < \rho < \rho_2\,, \\ J_2 = -I/(\pi\,\rho_3^2 - \pi\,\rho_2^2) \;\; \text{for} \;\; \rho_2 \le \rho \le \rho_3\,, \\ 0 \;\; \text{for} \;\; \rho > \rho_3\,. \end{cases}$$

   Now $\vec{B} = B_\phi\,\hat{\phi}$ with $B_\phi = B = |\vec{B}|$ and

   $$2\pi\,\rho\,B_\phi = \rho \int_0^{2\pi} d\phi\,B_\phi = \oint d\vec{s}\cdot\vec{B} = \int_S da\,\hat{a}\cdot(\nabla\times\vec{B}) = \frac{4\pi}{c}\int_S da\,\hat{a}\cdot\vec{J}$$

$\rho \le \rho_1$:

$$2\pi\,\rho\,B_\phi = \frac{4\pi}{c}\,J_1\,\pi\,\rho^2 \;\Rightarrow\; B = \frac{2}{c}\,J_1\,\pi\,\rho = \frac{2\,I\,\rho}{c\,\rho_1^2}\ .$$

$\rho_1 < \rho < \rho_2$:

$$2\pi\,\rho\,B_\phi = \frac{4\pi}{c}\,J_1\,\pi\,\rho_1^2 \;\Rightarrow\; B = \frac{2\,I}{c\,\rho}\ .$$

$\rho_2 \le \rho \le \rho_3$:

$$B = \frac{2\,I}{c\,\rho} + \frac{2\,J_2}{c\,\rho}\,\pi\,(\rho^2 - \rho_2^2) = \frac{2\,I}{c\,\rho}\,\frac{(\rho_3^2 - \rho^2)}{(\rho_3^2 - \rho_2^2)}\ .$$

Finally, $\rho > \rho_3$:

$$B = 0\ .$$

3. Potential from distinct BCs on half-spheres.

(a) We have to calculate

$$\Phi(r) = R^2 \int_S d\Omega'\,\Phi_0\,\frac{\partial G_D}{\partial r'}\bigg|_{r'=R}\ ,$$

where the surface is the upper half-sphere and the derivative of the Green function is given by

$$\frac{\partial G_D}{\partial r'}\bigg|_{r'=R} = \frac{R^2 - r^2}{4\pi\,R\,\left(R^2 + r^2 - 2Rr\,\cos\gamma\right)^{3/2}}$$

with $\cos\gamma = \cos(\theta)\,\cos(\theta') + \sin(\theta)\,\sin(\theta')\,\cos(\phi - \phi')$. For $r = z$ on the z-axis we have $\theta = 0$ and, hence, $\cos\gamma = \cos\theta'$. The integral over the upper half-sphere becomes

$$\Phi(z) = \Phi_0\,\frac{R}{2}\,(R+z)\,(R-z)\int_0^1 \frac{d\cos\theta'}{\left(R^2 + z^2 - 2R\,z\,\cos\theta'\right)^{3/2}}\ .$$

With $a = R^2 + z^2$ and $b = 2Rz$ the integral is

$$\begin{aligned}\int_0^1 \frac{dx}{(a - b\,x)^{3/2}} &= \frac{2}{b\,(a - b\,x)^{1/2}}\bigg|_0^1 = \frac{2}{b\,\sqrt{a-b}} - \frac{2}{b\,\sqrt{a}}\\ &= \frac{1}{Rz\,\sqrt{(R-z)^2}} - \frac{1}{Rz\,\sqrt{R^2+z^2}}\end{aligned}$$

and

$$\Phi(z) = \Phi_0\,\frac{1}{2z}\,(R+z)\left(1 - \frac{R-z}{\sqrt{R^2+z^2}}\right)\ .$$

(b) Special values:

$$\begin{aligned}
z &= -R &\Rightarrow \quad \Phi(-R) &= 0\,, \\
z &= -\frac{R}{2} &\Rightarrow \quad \Phi\left(-\frac{R}{2}\right) &= \Phi_0 \left(-\frac{1}{2}\right)\left(1-\frac{3}{\sqrt{5}}\right) = 0.17082\,\Phi_0 \\
z &= \frac{R}{2} &\Rightarrow \quad \Phi\left(\frac{R}{2}\right) &= \Phi_0 \left(\frac{3}{2}\right)\left(1-\frac{1}{\sqrt{5}}\right) = 0.82918\,\Phi_0 \\
z &= R &\Rightarrow \quad \Phi(R) &= \Phi_0 \frac{2R}{2R} = \Phi_0\,, \\
z &\to 0 &\Rightarrow \quad \Phi(0) &= \Phi_0 \frac{R}{2z}\left(1-\frac{R-z}{R}\right) \to \frac{1}{2}\,\Phi_0\,.
\end{aligned}$$

The plot is shown in the figure.

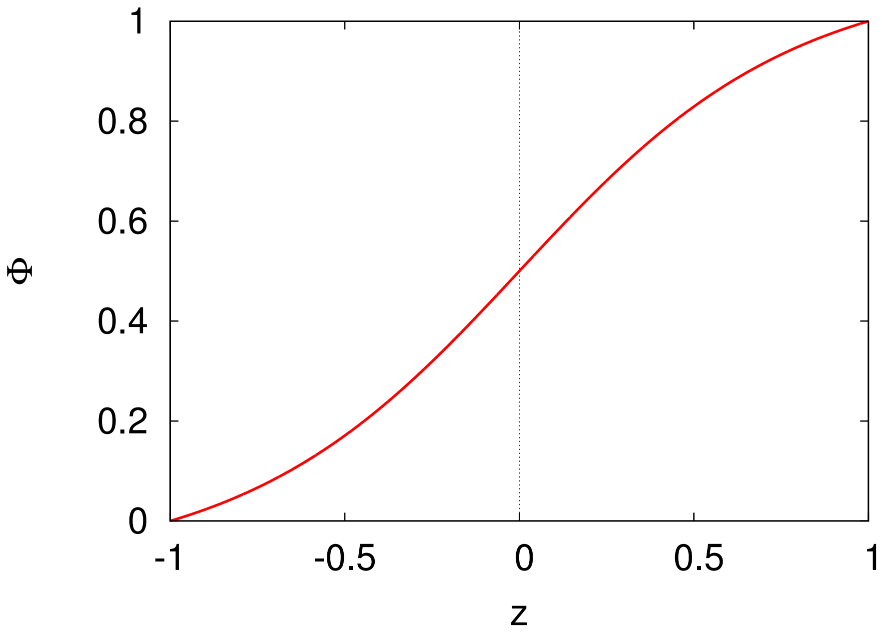

FIG. 1: Potential as function of z.