Electrodynamics A (PHY 5346) Fall 2016 Solutions Test on Homework December 1.

1. Time in Minkowski space (from HW 5, set 2).

   We use natural units, $c = 1$, and give all results in units of seconds.

   (a) It follows from $x^1 = 4\,t/5 = t - 15$ that the coordinates of B_0 are $(t, x) = (75, 60)$.

   (b) The proper time of B is then $\tau = \sqrt{t^2 - x^2} = 45$.

   (c) At position A_2 the time on the clock of A is $75 + 60 = 135$.

2. Potential in a rectangular box (HW 26, set 8).

   By separation of variables, we get

$$\Phi(x,y,z) = \sum_{nm} A_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sinh\left(z\pi\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}\right)$$

   The BC condition given in the problem is

$$\Phi(x,y,c) = \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi y}{b}\right) + \sin\left(\frac{3\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right) .$$

   From this we find that only $(n = 1, m = 2)$ and $(n = 3, m = 1)$ contribute to the expansion

$$\begin{aligned}\Phi(x,y,z) &= A_{12} \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi y}{b}\right) \sinh\left(z\pi\sqrt{\frac{1}{a^2}+\frac{4}{b^2}}\right) \\ &+ A_{31} \sin\left(\frac{3\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right) \sinh\left(z\pi\sqrt{\frac{9}{a^2}+\frac{1}{b^2}}\right) .\end{aligned}$$

   Comparing to

$$\Phi(x,y,c) = \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi y}{b}\right) + \sin\left(\frac{3\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right)$$

   we find

$$A_{31} = \frac{1}{\sinh\left(c\pi\sqrt{9/a^2+1/b^2}\right)} \quad \text{and} \quad A_{12} = \frac{1}{\sinh\left(c\pi\sqrt{1/a^2+4/b^2}\right)} .$$

   The same result is obtained by using the integral definitions of the coefficients and performing the integrations.

3. Dielectric sphere (HW 41, set 12).

(a) We expand the potential into spherical harmonics. Due to the axial symmetry we have only $m = 0$ contributions, which are Legendre polynomials.

$$\begin{aligned} \text{Inside}: \quad \Phi_{\text{in}} &= \sum_{l=0}^{\infty} A_l\, r^l\, P_l(\cos\theta)\,, \\ \text{Outside}: \quad \Phi_{\text{out}} &= \sum_{l=0}^{\infty} \left[B_l\, r^l + C_l\, r^{-(l+1)} \right] P_l(\cos\theta)\,. \end{aligned}$$

For $r \to \infty$: $\Phi \to -E_0\, z = -E_0\, r\, \cos\theta$. This implies that the only non-vanishing coefficient B_l is

$$B_1 = -E_0\ .$$

BCs at $r = R$:

$$\begin{aligned} \text{Tangential}: \quad \frac{1}{R}\, \frac{\partial \Phi_{\text{in}}}{\partial \theta}\bigg|_{r=R} &= \frac{1}{R}\, \frac{\partial \Phi_{\text{out}}}{\partial \theta}\bigg|_{r=R}\,, \\ \text{Normal}: \quad \epsilon_1\, \frac{\partial \Phi_{\text{in}}}{\partial r}\bigg|_{r=R} &= \epsilon_2\, \frac{\partial \Phi_{\text{out}}}{\partial r}\bigg|_{r=R}\,. \end{aligned}$$

(b) Matching for the tangential BC $\partial P_l / \partial\theta$ term by term (they are independent functions of θ), we find:

$$\begin{aligned} A_1\, R = B_1\, R + \frac{C_1}{R^2} \;\Rightarrow\; A_1 &= -E_0 + \frac{C_1}{R^3} \\ \text{and for} \quad l \geq 2: \quad A_l &= \frac{C_l}{R^{(2l+1)}}\ . \end{aligned}$$

Similarly we match for the normal BC P_l term by term and find:

$$\begin{aligned} \epsilon_1\, A_1 &= -\epsilon_2\, E_0 - 2\, \epsilon_2\, \frac{C_1}{R^3} \\ \text{and for} \quad l \geq 2: \quad \epsilon_2\, l\, A_l &= -(l+1)\, \epsilon_2\, \frac{C_l}{R^{(2l+1)}}\ . \end{aligned}$$

Putting the $l \geq 2$ equations together we get

$$\frac{C_l}{R^{(2l+1)}} = -\frac{(l+1)\, \epsilon_2}{l\, \epsilon_1}\, \frac{C_l}{R^{(2l+1)}} \;\Rightarrow\; C_l = 0, \quad l \geq 2 \;\Rightarrow\; A_l = 0, \quad l \geq 2\ .$$

Let us define

$$\epsilon = \frac{\epsilon_1}{\epsilon_2}\ .$$

With this notation we have

$$\epsilon\, A_1 = -E_0 - 2\, \frac{C_1}{R^3}\ .$$

Combining this with our other equation for A_1 gives

$$A_1 \;=\; -\left(\frac{3}{\epsilon+2}\right)\,E_0\,, \qquad C_1 \;=\; \left(\frac{\epsilon-1}{\epsilon+2}\right)\,R^3\,E_0\,.$$

Therefore,

$$\begin{aligned}
\Phi_{\rm in} \;&=\; -\left(\frac{3}{\epsilon+2}\right)\,E_0\,r\,\cos(\theta) \;=\; -\left(\frac{3}{\epsilon+2}\right)\,E_0\,z\,,\\
\Phi_{\rm out} \;&=\; -E_0\,z + \left(\frac{\epsilon-1}{\epsilon+2}\right)\,E_0\,\frac{R^3}{r^2}\,\cos(\theta)\,.
\end{aligned}$$

Note that the last term is the potential of a dipole. The electric fields are then

$$\begin{aligned}
\vec{E}_{\rm in} \;&=\; -\nabla\,\Phi_{\rm in} \;=\; \left(\frac{3}{\epsilon+2}\right)\,\vec{E}_0\,,\\
\vec{E}_{\rm out} \;&=\; -\nabla\,\Phi_{\rm out} \;=\; \vec{E}_0 - \left(\frac{\epsilon-1}{\epsilon+2}\right)\,R^3\,\left(\frac{r^2\,\vec{E}_0 - 3\vec{r}\,(\vec{r}\cdot\vec{E}_0)}{r^5}\right)\,.
\end{aligned}$$

(c) The surface charge density is

$$\begin{aligned}
\sigma_{\rm pol} \;&=\; \frac{1}{4\pi}\,(E^r_{\rm out} - E^r_{\rm in}) \;=\; \frac{1}{4\pi}\,\hat{r}\cdot\left(\vec{E}_{\rm out} - \vec{E}_{\rm in}\right)\\
&=\; \frac{1}{4\pi}\,E_0\,\cos(\theta)\,\left[\left(1 - \frac{3}{\epsilon+2}\right) - \left(\frac{\epsilon-1}{\epsilon+2}\right)\,(1-3)\right]\\
&=\; \frac{3}{4\pi}\,\left(\frac{\epsilon-1}{\epsilon+2}\right)\,E_0\,\cos(\theta)\;.
\end{aligned}$$