Electrodynamics A (PHY 5346) Fall 2016 Solutions Midterm October 11.

1. **Spacetrip.**

We only consider the first quarter (1 year) of the flight. The other results for t are the same due to symmetry. With $\beta = v/c$ the acceleration in the rest frame is given by

$$\frac{g}{c} = \frac{d\beta}{d\tau} = \frac{d\zeta}{d\tau}\,,$$

where ζ is the rapidity. As rapidities are additive, the following equation holds in the earth frame:

$$d\zeta(\tau) = \frac{d\zeta}{d\tau}\,d\tau = \frac{g}{c}\,d\tau\,.$$

With the initial condition $\zeta(0) = 0$ this integrates to

$$\zeta = \int_0^{\zeta} d\zeta' = \frac{g}{c}\int_0^{\tau} d\tau' = \frac{g}{c}\,\tau\,.$$

The age of the twin on earth follows from $dt' = \cosh(\zeta)\,d\tau'$:

$$\int_0^t dt' = t = \int_0^{\tau} \cosh[\zeta(\tau')]\,d\tau' = \int_0^{\tau} \cosh\left[\frac{g}{c}\,\tau'\right]\,d\tau' = \frac{c}{g}\,\sinh\left[\frac{g}{c}\,\tau\right]\,.$$

Inserting $\tau = 1\,\mathrm{year} = 365 \times 24 \times 3600\,[s]$, $c = 3 \times 10^8\,[m/s]$ and $g = 9.81\,[m/s^2]$, we find $t = 1.187$ years for a quarter of the trip and 4.748 years for the entire trip.

Distance traveled.

Seen from earth: maximum distance $= 2\,x_1$ with

$$\begin{aligned} x_1 &= \int_0^{t_1} v(t)\,dt = \int_0^{\tau_1} \cosh\left[\zeta(\tau)\right]\,v(\tau)\,d\tau \\ &= c\int_0^{\tau_1} \sinh\left(\frac{g\tau}{c}\right)\,d\tau \;=\; \frac{c^2}{g}\,\left[\cosh\left(\frac{g\tau_1}{c}\right) - 1\right]\,. \end{aligned}$$

Numerical values: $x_1 = 0.563$ light years, maximum distance $= 1.126$ light years.

2. **Lorentz invariance of anti-symmetry**

$$F'^{\alpha\beta} = a^{\alpha}{}_{\hat\alpha}\,a^{\beta}{}_{\hat\beta}\,F^{\hat\alpha\hat\beta}$$

$$F'^{\beta\alpha} = a^{\beta}{}_{\hat\alpha}\,a^{\alpha}{}_{\hat\beta}\,F^{\hat\alpha\hat\beta} = -a^{\beta}{}_{\hat\alpha}\,a^{\alpha}{}_{\hat\beta}\,F^{\hat\beta\hat\alpha} = -a^{\alpha}{}_{\hat\alpha}\,a^{\beta}{}_{\hat\beta}\,F^{\hat\alpha\hat\beta} = -F'^{\alpha\beta}\,.$$

3. **Some electromagnetic invariants.**

From the lecture notes (1.161), (1.162):

$$(F^{\alpha\beta}) = \begin{pmatrix} 0 & -E^x & -E^y & -E^z \\ E^x & 0 & -B^z & B^y \\ E^y & B^z & 0 & -B^x \\ E^z & -B^y & B^x & 0 \end{pmatrix}, \quad (F_{\alpha\beta}) = \begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & -B^z & B^y \\ -E^y & B^z & 0 & -B^x \\ -E^z & -B^y & B^x & 0 \end{pmatrix}$$

and, therefore,

$$F_{\alpha\beta}F^{\alpha\beta} = 2\left(\vec{B}^{\,2} - \vec{E}^{\,2}\right) .$$

Next, from (1.164):

$$({}^*F^{\alpha\beta}) = \begin{pmatrix} 0 & -B^x & -B^y & -B^z \\ B^x & 0 & E^z & -E^y \\ B^y & -E^z & 0 & E^x \\ B^z & E^y & -E^x & 0 \end{pmatrix} \quad ({}^*F_{\alpha\beta}) = \begin{pmatrix} 0 & B^x & B^y & B^z \\ -B^x & 0 & E^z & -E^y \\ -B^y & -E^z & 0 & E^x \\ -B^z & E^y & -E^x & 0 \end{pmatrix} .$$

and, therefore,

$$F_{\alpha\beta}\,{}^*F^{\alpha\beta} = -4\,\vec{E}\cdot\vec{B}\,, \quad {}^*F_{\alpha\beta}\,{}^*F^{\alpha\beta} = 2\left(\vec{E}^{\,2} - \vec{B}^{\,2}\right) .$$