3D

Cosmography of the surface of an expanding 4D sphere

$$\sum_{i=1}^{4} X_i(t)^2 = R(t)^2$$

$$\dot{R}(t) = \frac{dR}{dt} = H(t)\, R(t)$$

$H_0 = H(t_0)$, t_0 present time

Hubble constant

Orders of magnitude:

Example: $H_0 = \dfrac{1}{10^{10} \text{ years}}$.

Mega parsec: $10^6 \times 3.262$ light years.

H_0 often given as speed between two point at distance s:

Units:

$$[H_0] = \left[\frac{\text{distance / time}}{\text{distance}}\right] = \frac{1}{[\text{time}]}$$

$$H_0 \approx \frac{1}{3.17\times10^{16}\ s} = \frac{10^6\ pc}{3.17\times10^{16}\ s\cdot Mpc}$$

$$\approx \frac{98\ Km/s}{Mpc}$$

Measured values $\approx \frac{70\ Km/s}{Mpc}$

For distance of <u>One meter</u>:

$$H_0 = \frac{1}{10^{10}\ years} = \frac{10^{-10}\ m/year}{m}$$

0.1 nano meter per year.

Past time

<u>Let $t < t_0$:</u> Seperation of

<u>variable:</u> $\frac{dR}{dt} = HR \Rightarrow \frac{dR}{R} = H(t)\,dt$

$$\int_t^{t_0} \frac{dR}{R} = \ln R(t_0) - \ln R(t) =$$

$$\ln\left[\frac{R(t_0)}{R(t)}\right] = \int_t^{t_0} H(t')\,dt'$$

Present radius

Co 3

$$\frac{R(t_0)}{R(t)} = \frac{R_0}{R(t)} = \exp \int_t^{t_0} H(t')dt'$$

Spherical Coordinates: $R = R(t)$

Recall 3 b:

$$X_3 = R\cos(\theta_2) \qquad (\theta = \theta_2)$$

$$\rho = R\sin(\theta_2)$$

$$X_2 = \rho\sin(\theta_1) \qquad (\phi = \theta_1)$$

$$= R\sin(\theta_2)\sin(\theta_1)$$

$$X_1 = \rho\cos(\theta_1)$$

$$= R\sin(\theta_2)\cos(\theta_1)$$

$$d\vec{s} = \hat{\theta}_2 R\,d\theta_2 + \hat{\theta}_1 \rho\,d\theta_1$$

$(dr = 0$ on surface$)$

$$(ds)^2 = R^2 (d\theta_2)^2 + \rho^2 (d\theta_1)^2$$

$$= R^2\left[(d\theta_2)^2 + \sin^2(\theta_2)(d\theta_1)^2\right]$$

Extension to 4D:

$$x_4 = R\cos(\theta_3)$$

$$\rho_2 = R\sin(\theta_3)$$

$$x_3 = \rho_2 \cos(\theta_2) = R\sin(\theta_3)\cos(\theta_2)$$

$$\rho_1 = \rho_2 \sin(\theta_2) \quad [\rho_2 \text{ as } R \text{ in 3D}]$$

$$x_2 = \rho_1 \sin(\theta_1)$$

$$= R\sin(\theta_3)\sin(\theta_2)\sin(\theta_1)$$

$$x_1 = \rho_1 \cos(\theta_1)$$

$$= R\sin(\theta_3)\sin(\theta_2)\cos(\theta_1)$$

$$d\vec{s} = \hat{\theta}_3 R\,d\theta_3 + \hat{\theta}_2 \rho_2\,d\theta_2 + \hat{\theta}_1 \rho_1\,d\theta_1$$

$$|ds|^2 = R^2(d\theta_3)^2 + (\rho_2)^2(d\theta_2)^2 + (\rho_1)^2(d\theta_1)^2$$

$$= R^2\left[(d\theta_3)^2 + (\sin^2\theta_3)(d\theta_2)^2 + \sin^2\theta_3 \sin^2\theta_2 (d\theta_1)^2\right]$$

Robertson – Walker metric

Standard (Weinberg) notation

$(d\tau)^2 = (dt)^2 - (ds)^2$ with

$$(ds)^2 = R^2\left[\frac{(dr)^2}{1-Kr^2} + r^2(d\theta)^2 + r^2\sin^2(\theta)(d\phi)^2\right]$$

Suitable units: $K = +1, 0$ or -1.

Our case $K = +1$: [Preferred $K = 0$ flat universe.]

$r = \sin(\theta_3)$ $\searrow$

$\theta = \theta_2$, $r = S_2$ $\quad dr = \cos(\theta_3)\,d\theta_3$

$\phi = \theta_1$

$$(ds)^2 = R^2\left[\frac{\cos^2\theta_3\,(d\theta_3)^2}{1-\sin^2\theta_3} + (S_2)^2(d\theta_2)^2 + (S_1)^2(d\theta_1)^2\right]$$

agrees with our metric.

$k = 1$

Constant H universe:

$$\frac{R_0}{R(t)} = \exp\left[\underbrace{(t_0 - t)}_{>0} H\right]$$

with $t_0 = 0$, $t < 0$:

$$R(t) = R_0 \exp[Ht],$$

$$R(t) \to 0 \text{ for } t \to -\infty.$$

$t = 10^{10}$ years

$R(t) = e^{-1} R_0$

Redshift: (Light signals.) Emission at t, Observation at $t_0 = 0$

Assumption: wave stretches with expansion of universe.

$$\frac{R_0}{R(t)} = \frac{\lambda(t)}{\lambda_0}$$

$\lambda(t)$ ← Emitted at t, observed at t_0 on earth.

λ_0 ↑ observed in Lab experiment on earth.

$$\frac{R_0}{R(t)} = \exp[-Ht] > 1$$

NOT from relative motion!

So, we get t from the redshift

$$-Ht = \ln(\lambda(t)/\lambda_0)$$

$$t = -H^{-1} \ln\left[\frac{\lambda(t)}{\lambda_0}\right]$$

$$= -10^{10} \ln\left[\frac{\lambda(t)}{\lambda_0}\right] \text{ years}$$

<u>Redshift parameter</u>

$$z = \frac{\lambda(t) - \lambda_0}{\lambda_0} \quad \text{or} \quad \frac{\lambda(t)}{\lambda_0} = z+1$$

Observed up to $z < 2.5$

<u>$z = 2.5$ example:</u>

$$t = -10^{10} \ln(3.5) \approx -1.25 \times 10^{10} \text{ years}$$

$\simeq 12.5$ billion years

$H = 70 \Rightarrow \approx 17.5$ billion years

At What <u>distance</u> is the source corresponding to some redshift now?

Geodesic: (With coordinate chosen so that propagation is in $\hat{\theta}_3$ direction.)

Let $\theta = \theta_3$, $\theta_0 = \theta(t_0) = 0$: c speed of light

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{c}{R(t)} = \frac{c}{R_0} \exp[-Ht]$$

$$R(t) = R_0 \exp[Ht]$$

$$\theta(t) = \frac{c}{R_0} \int_0^t \exp(-Ht')\, dt' = -\frac{c}{R_0 H} \exp(-Ht')\Big|_0^t$$

$$= \frac{c}{R_0 H}\left[1 - \exp(-Ht)\right]$$

Future: $t \to +\infty$, $\theta_{max} = \theta(\infty) = \dfrac{c}{R_0 H}$

Past: $t \to -\infty$, $\theta(t) \to -\infty$

Distance of past source now: $(t < 0)$

$$s(t) = \theta(t) R_0 = \frac{c}{H}\left[1 - \exp(-Ht)\right]$$

Small t: $\theta(t) = \frac{c}{R_0 H}\left[1 - 1 + Ht + \dots\right]$

$\theta(t) = \frac{ct}{R_0}$, $\underline{s(t) = ct}$ speed of light

Distance and redshift: ($t < 0$, i.e., past)

$$S(t) = \frac{c}{H}\left[1 - \frac{R_0}{R(t)}\right] = \frac{c}{H}\left[1 - \frac{\lambda(t)}{\lambda_0}\right]$$

$$\underline{= -\frac{c}{H} z}, \quad z = \frac{\lambda(t)}{\lambda_0} - 1$$

Luminosity of a standard candle

$$\frac{L(t)}{L_0} = \frac{A_0}{A(t)}$$

(close-by area - fixed are covered at measurement.)

$$s(t) = R_0 \theta(t)$$

$$A(t) = 4\pi (R_0)^2 \sin^2\theta(t)$$
$$= 4\pi\, s(t)^2 \sin^2\theta(t) / \theta(t)^2$$

For small $\theta(t)$ (large R_0):

$$A(t) = 4\pi\, s(t)^2 = \frac{4\pi c^2}{H^2} z^2$$

$$\ln L(t) = \text{const} - 2 \ln z$$

$$L(t) \sim \frac{1}{z^2}$$