p. 47–53

Curl ①

Definition

$$\vec{\nabla} \times \vec{V} = \varepsilon_{ijk}\, \hat{x}_i \frac{\partial}{\partial x_j} V_k = \begin{vmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 \\ \partial_1 & \partial_2 & \partial_2 \\ V_1 & V_2 & V_2 \end{vmatrix}$$

$$\partial_i = \frac{\partial}{\partial x_i}$$

Product Rule:

$$\vec{\nabla} \times (f \vec{V}) = f\, \vec{\nabla} \times \vec{V} + (\vec{\nabla} f) \times \vec{V}$$

Examples: $\vec{\nabla} \times \vec{r} = \varepsilon_{ijk}\, \hat{x}_i \frac{\partial}{\partial x_j} x_k$

$$= \varepsilon_{ijk}\, \hat{x}_i\, \delta_{jk} = \varepsilon_{ijj}\, \hat{x}_i = 0$$

$$\nabla \times (\vec{r} f(r)) = f(r) \underbrace{\cancel{\vec{\nabla} \times \vec{r}}}_{0} + (\vec{\nabla} f) \times \vec{r}$$

$$= \frac{df}{dr}\, \hat{r} \times \vec{r} = 0$$

//

Curl of a ~~...~~ gradient:

$$\vec{\nabla} \times (\vec{\nabla} \cdot \phi) = \varepsilon_{ijk}\, \hat{x}_i \frac{\partial}{\partial x_j} \left(\frac{\partial \phi}{\partial x_k} \right)$$

$$= \varepsilon_{ijk}\, \hat{x}_i \frac{\partial}{\partial x_k} \frac{\partial \phi}{\partial x_j} = -\varepsilon_{ikj} \frac{\partial}{\partial x_k} \frac{\partial \phi}{\partial x_j}$$

$$= -\varepsilon_{ijk}\, \hat{x}_i \frac{\partial}{\partial x_j} \frac{\partial \phi}{\partial x_k} = -\vec{\nabla} \times (\vec{\nabla} \cdot \phi)$$

$$\Rightarrow \quad \underline{\underline{\vec{\nabla} \times (\nabla \phi) = 0}}$$

General:

$$A_{ij} = -A_{ji}$$
$$S_{ij} = S_{ji}$$
$$A_{ij} S_{ij} = 0$$

Vector identities:

$$\nabla \times (\vec{A} \times \vec{B}) = \varepsilon_{ijk}\, \hat{x}_i \frac{\partial}{\partial x_j} \varepsilon_{klm} A_l B_m$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\, \hat{x}_i \frac{\partial}{\partial x_j} A_l B_m$$

$$= \hat{x}_i \frac{\partial}{\partial x_j} A_i B_j - \hat{x}_i \frac{\partial}{\partial x_j} A_j B_i$$

$$= \hat{x}_i \left(\frac{\partial A_i}{\partial x_j}\right) B_j + \hat{x}_i A_i \left(\frac{\partial B_j}{\partial x_j}\right)$$

$$- \hat{x}_i \left(\frac{\partial A_j}{\partial x_j}\right) B_i - \hat{x}_i A_j \left(\frac{\partial B_i}{\partial x_j}\right)$$

$$= (\vec{B}\cdot\vec{\nabla})\vec{A} + \vec{A}(\nabla\cdot\vec{B})$$
$$- \vec{B}(\nabla\cdot\vec{A}) - (\vec{A}\cdot\vec{\nabla})\vec{B}$$

Book p. 52 (1.7.5)

→ QM angular momentum operator:

$$\vec{L} = \vec{r}\times\vec{p}, \quad \vec{p} = -i\vec{\nabla}$$

$$L_i = -i\varepsilon_{ijk}\left(x_j \frac{\partial}{\partial x_k}\right)$$

$$L_1 = -i(x_2\partial_3 - x_3\partial_2)$$

$$L_2 = -i(x_3\partial_1 - x_1\partial_3)$$

$$L_3 = -i(x_1\partial_2 - x_2\partial_1)$$

Book p. 49

Curl (4)

Circulation (show figure):

$$\left.\begin{aligned} &(1)\quad V_x\left(x_0+\tfrac{dx}{2},\, y_0\right)\,dx \\ &(3)\quad -V_x\left(x_0+\tfrac{dx}{2},\, y_0+dy\right)\,dx \end{aligned}\right\} -\frac{\partial V_x}{\partial y}\,dx\,dy$$

$$\left.\begin{aligned} &(2)\quad V_y\left(x_0+dx,\, y_0+\tfrac{dy}{2}\right)\,dy \\ &(4)\quad -V_y\left(x_0,\, y_0+\tfrac{dy}{2}\right)\,dy \end{aligned}\right\} +\frac{\partial V_y}{\partial x}\,dx\,dy$$

$$\text{Circulation}_{1234} = \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)dx\,dy$$

$$\text{Circulation per unit area} = \left.\nabla\times\vec{V}\right|_z$$

In x-y plane

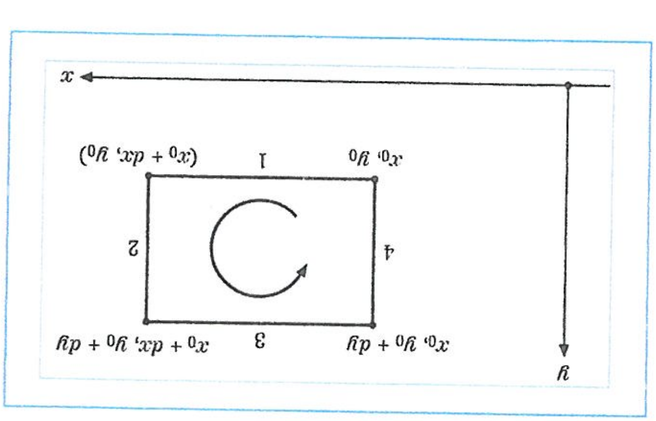

Figure 1.26
Circulation Around a Differential Loop

Successive Applications of $\vec{\nabla}$: (Book 1.8 p.53)

(a) $\vec{\nabla}\cdot\vec{\nabla}\varphi$, (b) $\vec{\nabla}\times\vec{\nabla}\varphi = 0$ (done)

(c) $\vec{\nabla}(\vec{\nabla}\cdot\vec{V})$ (d) $\vec{\nabla}\cdot(\nabla\times\vec{V})$

(e) $\vec{\nabla}\times(\vec{\nabla}\times\vec{V})$

(a) $$\vec{\nabla}\cdot\vec{\nabla}\varphi = \sum_i \hat{x}_i \partial_i \cdot \sum_j \hat{x}_j \partial_j \varphi$$

$$= \sum_i \sum_j \delta_{ij}\, \partial_i \partial_j \varphi = \sum_i \partial_i^2 \varphi$$

$$= \sum_i \frac{\partial^2 \varphi}{\partial x_i^2} = \nabla^2 \varphi$$

Laplace Operator

Laplacian of a radial function: (p.54)

$$\vec{\nabla}\cdot\vec{\nabla} g(r) = \vec{\nabla}\cdot\hat{r}\frac{dg}{dr} = \vec{\nabla}\cdot\frac{\vec{r}}{r}\frac{dg}{dr}$$

Product rule.

$$= (\vec{\nabla}\cdot\vec{r})\frac{1}{r}\frac{dg}{dr} + \vec{r}\cdot(\vec{\nabla} r^{-1})\frac{dg}{dr} + \hat{r}\cdot\vec{\nabla}\frac{dg}{dr}$$

$$= \frac{3}{r}\frac{dg}{dr} + \vec{r}\cdot\hat{r}\left(\frac{-1}{r^2}\right)\frac{d}{dr} + \hat{r}\cdot\hat{r}\,\frac{d^2 g}{dr^2}$$

$$= \frac{2}{r}\frac{dg}{dr} + \frac{d^2 g}{dr^2} \quad \text{(Book 1.8.1)}$$

(b) $\vec{\nabla}\times\vec{\nabla}\varphi = \varepsilon_{ijk}\,\hat{x}_i\,\partial_j\,\partial_k\,\varphi$

(again)

$= 0$

Because contraction of any asymmetric rank 2 (i.e. 2 indices) tensor with a symmetric rank 2 tensor is zero:

$$A_{ij} = -A_{ji}, \quad S_{ij} = S_{ji}$$

$$\underline{A_{ij}\,S_{ij}} = -A_{ji}\,S_{ji} =$$

$$-A_{kl}\,S_{kl} = \underline{-A_{ij}\,S_{ij}}$$

Backside

→ (d) $\vec{\nabla}\cdot\vec{\nabla}\times\vec{V} = \partial_i\,\varepsilon_{ijk}\,\partial_j V_k$

$= \varepsilon_{ijk}\,\partial_i\,\partial_j V_k = 0$ (symmetric w. anti-symmetric)

2b

Geometrical interpretation:

Flow of curl through all surfaces adds to zero.

But, NO pairwise cancellation.

Flow through up-down surfaces (show figure):

$$\hat{z}\,\frac{\partial}{\partial z}\cdot\vec{\nabla}\times\vec{V} = \frac{\partial}{\partial z}\,\vec{\nabla}\times\vec{V}\Big|_z$$

$$= \frac{\partial}{\partial z}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) \neq 0$$

$\vec{A}$ Vector potential

Example: $\vec{B} = \vec{\nabla} \times \vec{A}$ $\quad$ $\vec{B}$ magnetic field

$\vec{\nabla} \cdot \vec{B} = 0$ $\quad$ no magnetic monopoles.

Relation between (a),(c) and (e): (Book (1.16) p.55)

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{V}) = \varepsilon_{ijk}\, \varepsilon_{klm}\, \hat{x}_i\, \partial_j\, \partial_l\, V_m$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\, \hat{x}_i\, \partial_j\, \partial_l\, V_m$$

$$= \hat{x}_i\, \partial_j\, \partial_i\, V_j - \hat{x}_i\, \partial_j\, \partial_j\, V_i$$

$$= \vec{\nabla}(\vec{\nabla} \cdot \vec{V}) - \vec{\nabla}^2 \vec{V}$$

Example: (Book 1.8.2, p.56)

Electromagnetic waves from Maxwell's equations in vacuum

$$\vec{\nabla} \cdot \vec{B} = 0, \qquad \vec{\nabla} \cdot \vec{E} = 0 \quad \text{(no source)}$$

$$\vec{\nabla} \times \vec{B} = \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}, \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

From displacement current | Induction

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial}{\partial t} \vec{\nabla} \times \vec{B}$$

$$\vec{\nabla} \underbrace{\cancel{(\vec{\nabla} \cdot \vec{E})}}_{0} - \vec{\nabla}^2 \vec{E} = -\varepsilon_0 \mu_0 \frac{\partial^2}{\partial t^2} \vec{E}$$

$$\vec{\nabla}^2 \vec{E} = -\frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$ Wave equ.

$$\Rightarrow \quad c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$ speed of light with ε_0, μ_0 known from electrostatics.

Unification of optics and electrodynamics.

Come back to partial differential equs later.

> Addendum: $\hat{x}_j \times \hat{x}_k = \varepsilon_{ijk} \hat{x}_i$

Examples: $\hat{x}_1 \times \hat{x}_2 = \varepsilon_{312} \hat{x}_3 = \hat{x}_3$

$\hat{x}_2 \times \hat{x}_1 = \varepsilon_{321} \hat{x}_3 = -\hat{x}_3$

$\hat{x}_2 \times \hat{x}_3 = \varepsilon_{123} \hat{x}_1 = \hat{x}_1$

$\hat{x}_3 \times \hat{x}_1 = \varepsilon_{231} \hat{x}_2 = \hat{x}_2$