(Book p. 43 - 47)

$$\vec{\nabla} = \sum_{i=1}^{n} \hat{x}_i \frac{\partial}{\partial x_i}$$ Nabla Operator

Gradient: $\vec{\nabla} f(x_1, \dots, x_n)$ application to (scalar) function.

Divergence: $\vec{\nabla} \cdot \vec{V}$ application to vector function.

$$\vec{V} = \sum_{j=1}^{n} \hat{x}_j V_j (x_1, \dots, x_n)$$

$$\vec{\nabla} \cdot \vec{V} = \sum_i \sum_j \hat{x}_i \cdot \hat{x}_j \frac{\partial}{\partial x_i} V_j =$$

$$\sum_i \sum_j \delta_{ij} \frac{\partial V_j}{\partial x_i} = \sum_i \frac{\partial V_i}{\partial x_i}$$

Combination with scalar function (productrule):

$$\vec{\nabla} \cdot (f \vec{V}) = f (\vec{\nabla} \cdot \vec{V}) + \vec{V} \cdot (\vec{\nabla} f)$$

Proof: $$\sum_i \frac{\partial}{\partial x_i} (f V_i) = f \sum_i \left(\frac{\partial V_i}{\partial x_i}\right) + \sum_i V_i \frac{\partial f}{\partial x_i}$$

Applications:

$$\vec{\nabla}\cdot\vec{r} = \sum_i \frac{\partial x_i}{\partial x_i} = 3$$

$$\vec{\nabla}\cdot\vec{r}\,f(r) = f(r)\,(\nabla\cdot\vec{r}) + \vec{r}\cdot(\vec{\nabla} f)$$

$$= 3\,f(r) + \vec{r}\cdot\hat{r}\,\frac{df}{dr} = 3f(r) + r\frac{df(r)}{dr}$$

Examples: $\vec{r}\,f(r) = \vec{r}\,r^{n-1} = \hat{r}\,r^{n}$

$$\vec{\nabla}\cdot(\vec{r}\,r^{n-1}) = 3r^{n-1} + r\frac{d r^{n-1}}{dr}$$

$$= 3r^{n-1} + (n-1)\,r^{n-1} = \underline{\underline{(n+2)\,r^{n-1}}}$$

Electric field: $\vec{E} = \dfrac{q\,\hat{r}}{4\pi\varepsilon_0 r^2}$

Continuity Equ.:

Consider a (probability) density $\rho(x,y,z)$

with current $\vec{j} = \rho \vec{v}$.

Where the vector function $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$\vec{v}_i(x_i, y_i, z)$ is the local velocity.

Flow through $dy\,dz$ surface of parallelepiped

(show picture): $j_x\Big|_{x=dx} dy\,dz - j_x\Big|_{x=0} dy\,dz$
Book p. 45

$$j_x\Big|_{x=dx} = j_x\Big|_{x=0} + dx \frac{\partial j_x}{\partial x}$$

Net flow: $\dfrac{\partial j_x}{\partial x} dx\,dy\,dz$

The same with y and z direction:

$$\frac{\partial j_y}{\partial y} dx\,dy\,dz \,, \qquad \frac{\partial j_z}{\partial z} dx\,dy\,dz$$

Summing up: $\left(\dfrac{\partial j_x}{\partial x} + \dfrac{\partial j_y}{\partial y} + \dfrac{\partial j_z}{\partial z} \right) dx\,dy\,dz =$

$$\vec{\nabla} \cdot \vec{f} = \vec{\nabla} (\rho \vec{v})$$

$$= \underline{\underline{(\vec{v} \cdot \vec{\nabla})\rho}} + \rho \underbrace{\cancel{(\vec{\nabla} \cdot \vec{v})}}_{=0}$$

because: $\vec{\nabla} \cdot \vec{v} = \sum_i \frac{\partial \dot{x}_i}{\partial x} = \sum_i \frac{\partial^2 x}{\partial t \partial x}$

$$= \frac{\partial}{\partial t} \sum_i \frac{\partial x}{\partial x} = \frac{\partial}{\partial t} 3 = \underline{\underline{0}}$$

<u>Incompressible density:</u>

$$\rho(x(t), y(t), z(t), t)$$

$$0 = \frac{d\rho}{dt} = \sum_i \frac{\partial \rho}{\partial x_i} \frac{\partial x_i}{\partial t} + \frac{\partial \rho}{\partial t}$$

$$= \vec{\nabla} (\rho \vec{v}) + \frac{\partial \rho}{\partial t}$$

$$= \vec{\nabla} \vec{f} + \frac{\partial \rho}{\partial t}$$