Final Review: δ_{ij}, $\vec{a}\cdot\vec{b}$

$\varepsilon_{i_1 \dots i_n}$ / ε_{ijk}

Vector analysis:

$\vec{a}\times\vec{b}$, $\vec{a}\times(\vec{b}\times\vec{c}) \longleftrightarrow \varepsilon_{ijk}\varepsilon_{lmk}$

$\vec{\nabla} = \sum_i \hat{x}_i \frac{\partial}{\partial x_i}$, $\vec{\nabla}\phi$, $\vec{\nabla}\cdot\vec{A} =$

$\vec{\nabla}\times\vec{A} =$

Vector integration

Gauss' theorem: V with closed surface

$$\int_V \vec{\nabla}\cdot\vec{A}\, d^3x = \oint_S \vec{A}\cdot d\vec{\sigma}$$

Stoke's theorem: Surface with boundary C

$$\int_S (\nabla\times\vec{A})\cdot d\vec{\sigma} = \oint_C \vec{A}\cdot d\vec{s}$$

$\vec{A} = u\vec{\nabla}v$ (u, v scalar fcts).

Green's identity

$$\oint_S (u \vec{\nabla} v) d\vec{\sigma} = \int_V d^3x \left[u \vec{\nabla}^2 v + (\vec{\nabla} u)(\vec{\nabla} v) \right]$$

$$\parallel$$

$$\oint_S u \frac{\partial v}{\partial n} da \quad \Big| \quad u = v = U$$

$$U = \Phi_1 - \Phi_2, \quad \vec{\nabla}^2 U = 0$$

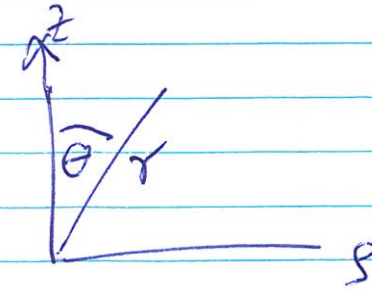
$$\oint_S U \frac{\partial U}{\partial n} da = \int_V |\vec{\nabla} U|^2 d^3x$$

$= 0$ with Dirichlet BC $\Phi_2 = \Phi_1$ on S

or Neumann BC $\frac{\partial \Phi_2}{\partial n} = \frac{\partial \Phi_1}{\partial n}$ on S.

Cylindrical Coordinates,

Spherical Coordinates:

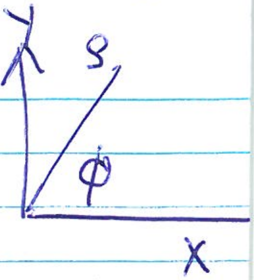


$X_3 = z = r \cos\theta$

$\rho = r \sin\theta$

F 3

$$x_2 = y = \rho\sin\phi = r\sin\theta\sin\phi$$
$$x_1 = x = \rho\cos\phi = r\sin\theta\cos\phi$$



$$d\vec{s} = \hat{r}\,dr + \hat{\theta}\,r\,d\theta + \hat{\phi}\,r\sin\theta\,d\phi$$

New locally orthonormal unit vectors.

Therefore,

$$|ds|^2 = (dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta\,(d\phi)^2$$
$$d^3x = dr\,(r\,d\theta)\,(r\sin\theta\,d\phi)$$
$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial\phi}$$

Directions of new unit vectors depend on position and, therefore, on time (when position depends on time).

Unlike Cartesian unit vectors!

$$
\begin{array}{c c}
 & \begin{array}{ccc} \qquad \hat{x} \qquad & \qquad \hat{y} \qquad & \qquad \end{array} \\
\begin{array}{c} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{array} &
\begin{pmatrix}
\sin\theta \cos\phi & \sin\theta \sin\phi & \cos\theta \\
+\cos\theta \cos\phi & +\cos\theta \sin\phi & -\sin\theta \\
-\sin\phi & \cos\phi & 0
\end{pmatrix}
\end{array}
$$

Matrix of expansion coefficients.

Some Work: $\vec{\nabla} \times A$

$\vec{\nabla}^2 \Phi$

in spherical coordinates.

Dirac delta fct.:

$$\delta\left[f(x)\right] = \sum_i \frac{\delta(x - x_i^0)}{|f'(x_i^0)|}$$

x_i^0 are the simple (!) zeros of $f(x)$.

Matrices:

Inhomogeneous linear equs.:

$A\vec{x} =$ F 5

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

Solution for $\Delta = \det|A| \neq 0$.

Cramer's Law:

$$x_k = \frac{\sum_i \Delta_{ik} b_i}{\Delta}, \quad \Delta_{ik} \text{ Co-factors.}$$

Non-Trivial

Homogeneous eqns.: $A\vec{x} = 0$

Solution for $\det|A| = 0$.

Only ratios x_j / x_i determined (if $x_i \neq 0$) and rank $n-1$.

Add/subtract equations!

Important in QM:

Eigenvalue eqns

$$H v = \lambda \hat{v}, \quad |\hat{v}|^2 = 1$$

$$\det |H - \lambda 1_{n \times n}| = 0$$

Characteristic Polynomial

$$\lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0 = 0$$

Symmetric $\tilde{H} = H^{-1}$

$\Rightarrow$ Real solutions.

If $\lambda_i \neq \lambda_j$ for all $i \neq j$

$\Rightarrow$ Eigenvectors $\hat{v}_i$ orthonormal

$$\hat{v}_i \cdot \hat{v}_j = \delta_{ij}$$