Jacobian Determinant

q_i general curvilinear coordinates $i = 1, \ldots, n$. To proof

for nD volume element:

$$\begin{vmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_1}{\partial q_2} & \ldots & \frac{\partial x_1}{\partial q_n} \\ \frac{\partial x_2}{\partial q_1} & \frac{\partial x_2}{\partial q_2} & \ldots & \frac{\partial x_2}{\partial q_n} \\ \vdots & & & \\ \frac{\partial x_n}{\partial q_n} & \frac{\partial x_n}{\partial q_2} & \ldots & \frac{\partial x_n}{\partial q_n} \end{vmatrix} dq_1 \ldots dq_n$$

$$= dx_1 \ldots dx_n = d^n X$$

Jacobian Determinat.

Proper generalitation of 1D

$$dx = \frac{dx}{dq} dq .$$

2 D case: $d^2x = dx_1\, dx_2$

Why is substituting

$$dx_1 = \frac{\partial x_1}{\partial q_1} dq_1 + \frac{\partial x_1}{\partial q_2} dq_2$$

$$dx_2 = \frac{\partial x_2}{\partial q_1} dq_1 + \partial x_2$$

NOT correct?

$d^2x = dx_1\, dx_2$ is a symbol for an infinitesimal area and NOT the product of individual dx_1 and dx_2.

In some part of mathematics (differential forms) a notation $dx_1 \wedge dx_2$ is therefore used. We can do without.

Consider the position vector in the x-y plane as function of q_1, q_2:

$$\vec{s} = \begin{pmatrix} x_1(q_1, q_2) \\ x_2(q_1, q_2) \end{pmatrix} = x_1(q_1, q_2)\,\hat{x}_1 + x_2(q_1, q_2)\,\hat{x}_2$$

<u>Infinitesimal changes:</u>

$$d\vec{q}_1 = \frac{\partial \vec{s}}{\partial q_1}\, dq_1 = \begin{pmatrix} \frac{\partial x_1}{\partial q_1} \\ \frac{\partial x_2}{\partial q_1} \end{pmatrix} dq_1$$

$$d\vec{q}_2 = \frac{\partial \vec{s}}{\partial q_2}\, dq_2 = \begin{pmatrix} \frac{\partial x_1}{\partial q_2} \\ \frac{\partial x_2}{\partial q_2} \end{pmatrix} dq_2$$

$$\hat{x}_3 \cdot \left(d\vec{q}_1 \times d\vec{q}_2 \right) = \hat{x}_3 \cdot \varepsilon_{3ij}\, \hat{x}_3 \frac{\partial x_i}{\partial q_1} \frac{\partial x_j}{\partial q_2}\, dq_1\, dq_2$$

$$= \left(\frac{\partial x_1}{\partial q_1} \frac{\partial x_2}{\partial q_2} - \frac{\partial x_2}{\partial q_1} \frac{\partial x_1}{\partial q_2} \right) dq_1\, dq_2$$

$$= \varepsilon_{ij} \frac{\partial x_i}{\partial q_1} \frac{\partial x_j}{\partial q_2}\, dq_1\, dq_2$$

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$$= \begin{vmatrix} \frac{\partial x_1}{\partial a_1} & \frac{\partial x_2}{\partial q_1} \\ \frac{\partial x_1}{\partial a_2} & \frac{\partial x_2}{\partial q_2} \end{vmatrix} \, da_1 \, da_2$$

Remember, definition of the determinant:

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \quad \text{matrix.}$$

$$\det A = |A| = \varepsilon_{i_1, \dots, i_n} a_{1 i_1} \dots a_{n i_n}$$

Proof:

$$\stackrel{!}{=} \varepsilon_{i_1, \dots, i_n} a_{i_1 1} \dots a_{i_n n}$$

For fixed $\left\{ \begin{matrix} \text{permutation} \\ i_1, \dots, i_n \end{matrix} \right\}$ let $\left\{ \begin{matrix} \text{permutation} \\ j_1 \dots j_n \end{matrix} \right\}$ be

$$a_{1 i_1} \dots a_{n i_n} = a_{j_1 1}, \dots, a_{j_n n}$$

To show $\varepsilon_{i_1 \dots i_n} = \varepsilon_{j_1 \dots j_n}$

i.e. ±1 match. To see this,

build both starting with $\varepsilon_{12\ldots(n-1)n}$:

$$\varepsilon_{2\ldots i_{j_1}\ldots n} = \varepsilon_{j_1 2 \ldots n}$$

$i_{j_1} = 1$

At position j_1 Number j_1

Because # of transposition agrees.

Next, either

$$\varepsilon_{3\ldots i_{j_2}\ldots i_{j_1}\ldots n} = \varepsilon_{j_1 j_2 \ldots n}$$

Or Number of transpositions agree.

$$\varepsilon_{3\ldots i_{j_1}\ldots i_{j_2}\ldots n} = \varepsilon_{j_1 j_2 \ldots n}$$

Position of number 2. Number of position

And so on for $i_{j_3} \ldots i_{j_n}$.

Therefore, also

$$\begin{vmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_2}{\partial q} & \cdots & \frac{\partial x_n}{\partial q_1} \\ \frac{\partial x_1}{\partial q_2} & \frac{\partial x_2}{\partial q_2} & \cdots & \frac{\partial x_n}{\partial q_2} \\ \vdots & & & \vdots \\ \frac{\partial x_1}{\partial q_n} & \frac{\partial x_2}{\partial q_n} & \cdots & \frac{\partial x_n}{\partial q_n} \end{vmatrix} dq_1 \ldots dq_n = d^n X$$

Infinitesimal area element in 3D space:

$$\vec{r} = \begin{pmatrix} x_1 (q_1, q_2) \\ x_2 (q_1, q_2) \\ x_3 (q_1, q_2) \end{pmatrix} = \sum_{i=1}^{3} x_i (q_1, q_2) \hat{x}_i$$

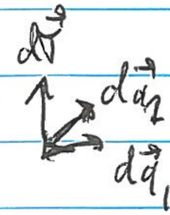
Infinitesimal changes:

$$d\vec{q}_1 = \frac{\partial \vec{r}}{\partial q_1}\, dq_1 = \begin{pmatrix} \dfrac{\partial x_1}{\partial q_1} \\[2ex] \dfrac{\partial x_2}{\partial q_2} \\[2ex] \dfrac{\partial x_3}{\partial q_3} \end{pmatrix} dq_1$$

$$d\vec{q}_2 = \frac{\partial \vec{r}}{\partial q_2}\, dq_2 = \begin{pmatrix} \dfrac{\partial x_1}{\partial q_2} \\[2ex] \dfrac{\partial x_2}{\partial q_2} \\[2ex] \dfrac{\partial x_3}{\partial q_2} \end{pmatrix} dq_2$$

$$d\vec{\sigma} = \varepsilon_{ijk}\, \hat{x}_i \left(\frac{\partial x_j}{\partial q_1}\right)\left(\frac{\partial x_k}{\partial q_2}\right) dq_1\, d_2$$

$$= \begin{vmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 \\[1ex] \dfrac{\partial x_1}{\partial q_1} & \dfrac{\partial x_2}{\partial q_1} & \dfrac{\partial x_3}{\partial q_1} \\[2ex] \dfrac{\partial x_1}{\partial q_2} & \dfrac{\partial X_2}{\partial q_2} & \dfrac{\partial x_3}{\partial q_2} \end{vmatrix} dq_1\, dq_2$$



Infinitesimal 3D volume element:

$$d\tau = d^3x = d\vec{q}_3 \cdot d\vec{\sigma} = \varepsilon_{ijk}\left(\frac{\partial x_i}{\partial q_3}\right)\left(\frac{\partial x_j}{\partial q_1}\right)\left(\frac{\partial x_k}{\partial q_2}\right)$$

Recall (!?): $\vec{c}\cdot(\vec{a}\times\vec{b})$ parallelepiped | $dq_1\, dq_2\, dq_3$

Because,

$$d\vec{q}_3 = \begin{pmatrix} \frac{\partial x_1}{\partial q_3} \\ \frac{\partial x_2}{\partial q_3} \\ \frac{\partial x_3}{\partial q_3} \end{pmatrix} dq_3$$

$$d^3x = d\tau = \varepsilon_{ijk}\left(\frac{\partial x_i}{\partial q_1}\right)\left(\frac{\partial x_j}{\partial q_2}\right)\left(\frac{\partial x_k}{\partial q_3}\right) dq_1\, dq_2\, dq_3$$

Due to cyclic permutation. As determinant

$$d\tau = d^3x = \begin{vmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_2}{\partial q_1} & \frac{\partial x_3}{\partial q_1} \\ \frac{\partial x_1}{\partial q_2} & \frac{\partial x_2}{\partial q_2} & \frac{\partial x_3}{\partial q_2} \\ \frac{\partial x_1}{\partial q_3} & \frac{\partial x_2}{\partial q_3} & \frac{\partial x_3}{\partial q_3} \end{vmatrix} dq_1\, dq_2\, dq_3$$

3D volume element in 4D

space: $\hat{x}_i$, $i = 1, 2, 3, 4$.

$$d\vec{\tau} = \varepsilon_{i_1 i_2 i_3 i_4}\, \hat{x}_{i_1} \left(\frac{\partial x_{i_2}}{\partial q_1}\right)\left(\frac{\partial x_{i_3}}{\partial q_2}\right)\left(\frac{\partial x_{i_4}}{\partial q_3}\right) dq_1\, dq_2\, dq_3$$

4D volume element:

$$d^4\tau = d^4x = \varepsilon_{i_1 i_2 i_3 i_4} \left(\frac{\partial x_{i_1}}{\partial q_1}\right)\left(\frac{\partial x_{i_2}}{\partial q_2}\right)\left(\frac{\partial x_{i_3}}{\partial q_3}\right)\left(\frac{\partial x_{i_4}}{\partial q_4}\right) dq_1\, dq_2\, dq_3\, dq_4$$

$$= d\vec{q}_4 \cdot d\vec{\tau}$$

$d^5\tau = d\vec{q}_5 \cdot d^4\vec{\tau}$ etc to nD.