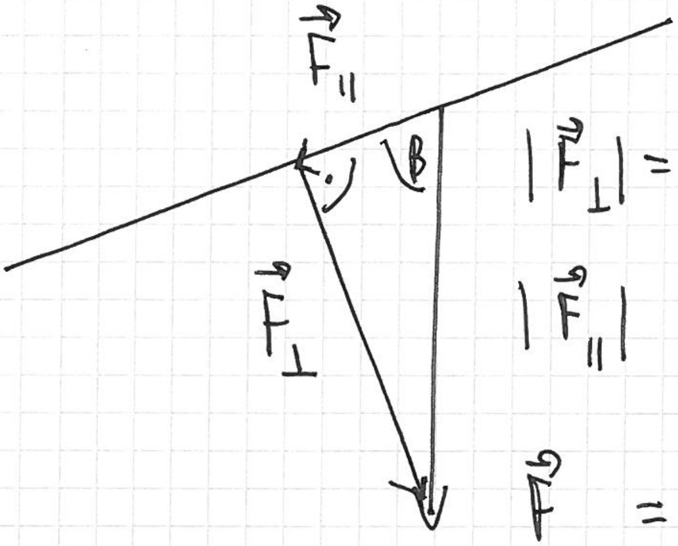
Decomposition of vectors

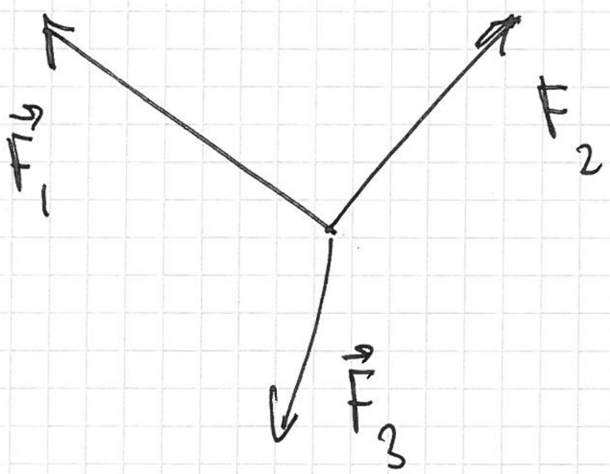


$$|\vec{F}_{\perp}| = F_{\perp} = F \cos\beta$$

$$|\vec{F}_{\parallel}| \quad F_{\parallel} = F \sin\beta$$

$$\vec{F} = -F\hat{z}$$

Static equilibrium:



etc.

$$\sum \vec{F}_i = 0$$

What should you memorize?

Basic definitions and operations.
NOT results.

$$\delta_{ij} = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases}$$ Kronecker delta

Permutation $i_1, \dots, i_n$: Rearrangement of number $1, \dots, n$ called **odd** when obtained by an odd number of transpositions, **even** when obtained by an even number of transpositions from $1, \dots, n$.

Transposition:
Interchange of two neighbouring elements of a permutation.

Levi-Civita Tensor:

$$\varepsilon_{i_1, \dots, i_n} = \begin{cases} 1 & \text{for even permutation of } 1, \dots, n \\ -1 & \text{for odd permutation of } 1, \dots, n \\ 0 & \text{otherwise} \end{cases}$$

Totally antisymmetric tensor

Cyclic permutations in 3 D

$$\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij} =$$

$$- \varepsilon_{ikj} = - \varepsilon_{kji} = - \varepsilon_{jik}$$

Cartesian unit vectors:

$$\hat{x}_i \cdot \hat{x}_j = \delta_{ij}$$

General vector:

$$\vec{a} = \sum_{i=1}^{n} a_i \hat{x}_i = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

Dot or scalar product of two vectors

$$\vec{a} \cdot \vec{b} = \sum_{i=1}^{n} a_i \hat{x}_i \cdot \sum_{j=1}^{n} b_j \hat{x}_j =$$

$$\sum_i \sum_i a_i b_j \, \hat{x}_i \cdot \hat{x}_j = \sum_i \sum_j a_i b_j \delta_{ij}$$

$$= \sum_i a_i b_i$$

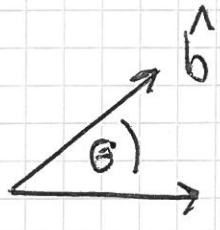
Length of a vector $a = |\vec{a}| = \sqrt{\vec{a}\cdot\vec{a}}$

Unit vector in direction of $\vec{a}$:

$$\hat{a} = \frac{\vec{a}}{a}\,, \qquad \hat{a}\cdot\hat{a} = 1$$

Angle between two vectors (definition of the cosine function):

$$\cos\theta = \hat{a}\cdot\hat{b}$$



3D Levi-Civita identity

$$\sum_i \varepsilon_{ijk}\,\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

Vector product:

$$\vec{a}\times\vec{b} = \sum_i\sum_j\sum_k \hat{x}_i\, a_j\, b_k$$

$$|\vec{a}\times\vec{b}| = ab\sin\theta$$

Proof: Use the 3D Levi-Civita identity to expand $(\hat{a}\times\hat{b})\cdot(\hat{a}\times\hat{b})$ into scalar vector products to find
$(\hat{a}\times\hat{b})\cdot(\hat{a}\times\hat{b}) = 1-\cos^2\theta = \sin^2\theta$.

Einstein Summation Convention

Two identical indices are automatically summed.

Examples: $\vec{a} \cdot \vec{b} = a_i b_i$

$$\vec{a} \times \vec{b} = \varepsilon_{ijk} \hat{x}_i a_j b_k$$

$$\varepsilon_{ijk} \varepsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}$$

$$\delta_{ij} a_i b_j = a_i b_i = \vec{a} \cdot \vec{b}$$

Nabla Operator:

$$\vec{\nabla} = \sum_i \hat{x}_i \frac{\partial}{\partial x_i} = \sum_i \hat{x}_i \partial_i \quad \text{with } \partial_i = \frac{\partial}{\partial x_i}$$

With Einstein convention: $\vec{\nabla} = \hat{x}_i \partial_i$

$\vec{r} = \hat{x}_i x_i$ position vector

$r = \sqrt{x_i x_i}$ magnitude of r

$= \sqrt{\sum_i x_i^2}$

Gradient: $\vec{\nabla} f$

Application of $\vec{\nabla}$ to a scalar function $f(x_1, \dots, x_n)$.

Examples:

$$\vec{\nabla} r = \hat{x}_i \partial_i \sqrt{x_j x_j} = \frac{\hat{x}_i \partial_i (x_j x_j)}{2\sqrt{x_j x_j}}$$

Chain rule

$$= \frac{2 \hat{x}_i \delta_{ij} x_j}{2 r} = \frac{\cancel{2} \hat{x}_i x_i}{\cancel{2} r} = \frac{\vec{r}}{r} = \hat{r}$$

Use $\partial_i x_j = \delta_{ij}$ and product rule.

$$\vec{\nabla} f(r) = \hat{x}_i \partial_i f(r) = \hat{x}_i \frac{df}{dr} \partial_i r$$

$$= \frac{df}{dr} \hat{x}_i \partial_i r = \frac{df}{dr} \hat{r}$$

Chain rule again

Divergence: $\vec{\nabla} \cdot \vec{V}$

Application of $\vec{\nabla}$ to a vector function $\vec{V} = \hat{x}_i V_i$, $V_i = V_i(x_1, \ldots x_n)$.

Examples: (derive!) $\vec{\nabla} \cdot \vec{r} = 3$

$$\vec{\nabla} \cdot \vec{r} f(r) = 3 f(r) + \vec{r} \cdot \underbrace{\vec{\nabla} f(r)}_{\hat{r} \frac{df}{dr}}$$

$$= 3 f(r) + r \frac{df}{dr}$$

$$\vec{\nabla} \cdot \vec{r}\, r^{n-1} = (n+2) r^{n-1}$$

$$= 0 \text{ for } n = -2,\ r \neq 0.$$

Curl: $\vec{\nabla} \times \vec{V} = \varepsilon_{ijk} \hat{x}_i \partial_j V_k$

with $\partial_j = \frac{\partial}{\partial x_j}$

Examples: (derive!)

$$\vec{\nabla} \times \vec{r} = \varepsilon_{ijk} \hat{x}_i \partial_j x_k = \varepsilon_{ijk} \hat{x}_i \delta_{jk}$$

$$= \varepsilon_{ijj} \hat{x}_i = \underline{\underline{0}}$$

$$\vec{\nabla} \times \vec{\nabla} f = \varepsilon_{ijk} \hat{x}_i \partial_j \partial_k f = 0$$

$$\vec{\nabla} \times (\vec{A} \times \vec{B}) = \varepsilon_{ijk} \hat{x}_i \partial_j \varepsilon_{klm} A_l B_m$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \hat{x}_i \partial_j A_l B_m$$

$$= \hat{x}_i \partial_j A_i B_j - \hat{x}_i \partial_j A_j B_i$$

$$= B_j \partial_j \vec{A} + \vec{A}\, \vec{\nabla} \cdot B - A_j \partial_j \vec{B} - \vec{B}\, \vec{\nabla} \cdot \vec{A}$$

$$= \underline{\underline{(\vec{B} \cdot \vec{\nabla})\vec{A} + \vec{A}\, \vec{\nabla} \cdot B - (\vec{A} \cdot \vec{\nabla})\vec{B} - \vec{B}\, \vec{\nabla} \cdot \vec{A}}}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{V}) = \varepsilon_{ijk} \hat{x}_i \partial_j \varepsilon_{klm} \partial_l V_m$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \hat{x}_i \partial_j \partial_l V_m$$

$$= \hat{x}_i \partial_i \partial_j V_j - \partial_j \partial_j \hat{x}_i V_i$$

$$\underline{\underline{= \vec{\nabla} (\vec{\nabla} \cdot \vec{V}) - \vec{\nabla}^2 \vec{V}}}$$

and so on.