Mathematical Physics — PHZ 3113
Integral Definitions of Gradient, Divergence and Curl (February 11, 2013)

The case of rectangular volume

$$V = \lim_{V\to 0} \int_V d\tau = \lim_{V\to 0} \int_V dx\, dy\, dz \quad (1)$$

is considered. S is the surface of V.

Gradient (Book p.66):

$$\nabla\phi \;=\; \lim_{V\to 0} \frac{\int_S \phi d\vec{\sigma}}{\int_V d\tau} \qquad (2)$$

$$= \; [\phi(x,y,z+dz) - \phi(x,y,z)]\; \frac{dx\, dy\, \hat{z}}{dx\, dy\, dz}$$

$$+ \; [\phi(x+dx,y,z) - \phi(x,y,z)]\; \frac{dy\, dz\, \hat{x}}{dx\, dy\, dz}$$

$$+ \; [\phi(x,y+dy,z) - \phi(x,y,z)]\; \frac{dz\, dx\, \hat{y}}{dx\, dy\, dz}$$

$$= \; \frac{\partial\phi}{\partial z}\, dz\, \frac{dx\, dy\, \hat{z}}{dx\, dy\, dz} + \frac{\partial\phi}{\partial x}\, dx\, \frac{dy\, dz\, \hat{x}}{dx\, dy\, dz}$$

$$+ \; \frac{\partial\phi}{\partial y}\, dy\, \frac{dz\, dx\, \hat{y}}{dx\, dy\, dz} = \frac{\partial\phi}{\partial z}\,\hat{z} + \frac{\partial\phi}{\partial x}\,\hat{x} + \frac{\partial\phi}{\partial y}\,\hat{y}\,.$$

Divergence (Book p.66):

$$\nabla \cdot \vec{A} = \lim_{V \to 0} \frac{\int_S \vec{A} \cdot d\vec{\sigma}}{\int_V d\tau} \tag{3}$$

$$= [A_z(x, y, z + dz) - A_z(x, y, z)] \frac{dx\,dy}{dx\,dy\,dz}$$

$$+ [A_x(x + dx, y, z) - A_x(x, y, z)] \frac{dy\,dz}{dx\,dy\,dz}$$

$$+ \left[A_y(x, y + dy, z) - A_y(x, y, z)\right] \frac{dz\,dx}{dx\,dy\,dz}$$

$$= \frac{\partial A_z}{\partial z} dz \frac{1}{dz} + \frac{\partial A_x}{\partial x} dx \frac{1}{dx} + \frac{\partial A_y}{\partial y} dy \frac{1}{dy}$$

$$= \frac{\partial A_z}{\partial z} + \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} .$$

Curl (Book p.66):

$$\nabla \times \vec{A} = \lim_{V \to 0} \frac{\int_S d\vec{\sigma} \times \vec{A}}{\int_V d\tau} \tag{4}$$

$$
\begin{aligned}
& d\vec{\sigma} \times \vec{A} \ = \\
& dx\, dy\, \hat{z} \times \left[\vec{A}(x,y,z+dz) - \vec{A}(x,y,z) \right] \\
+ \ & dy\, dz\, \hat{x} \times \left[\vec{A}(x+dx,y,z) - \vec{A}(x,y,z) \right] \\
+ \ & dz\, dx\, \hat{y} \times \left[\vec{A}(x,y+dy,z) - \vec{A}(x,y,z) \right] \\
= \ & dx\, dy\, \hat{z} \times \frac{\partial \vec{A}}{\partial z}\, dz + dy\, dz\, \hat{x} \times \frac{\partial \vec{A}}{\partial x}\, dx \\
+ \ & dz\, dx\, \hat{y} \times \frac{\partial \vec{A}}{\partial y}\, dy
\end{aligned}
$$

$$
\begin{aligned}
& \frac{d\vec{\sigma} \times \vec{A}}{dx\, dy\, dz} = \hat{z} \times \frac{\partial \vec{A}}{\partial z} + \hat{x} \times \frac{\partial \vec{A}}{\partial x} + \hat{y} \times \frac{\partial \vec{A}}{\partial y} \\
= \ & \hat{y}\, \frac{\partial A_x}{\partial\, z} - \hat{x}\, \frac{\partial A_y}{\partial\, z} + \hat{z}\, \frac{\partial A_y}{\partial\, x} - \hat{y}\, \frac{\partial A_z}{\partial\, x} \\
+ \ & \hat{x}\, \frac{\partial A_z}{\partial\, y} - \hat{z}\, \frac{\partial A_x}{\partial\, y} \\
= \ & \hat{x}\, \left(\frac{\partial A_z}{\partial\, y} - \frac{\partial A_y}{\partial\, z} \right) + \hat{y}\, \left(\frac{\partial A_x}{\partial\, z} - \frac{\partial A_z}{\partial\, x} \right) \\
+ \ & \hat{z}\, \left(\frac{\partial A_y}{\partial\, x} - \frac{\partial A_x}{\partial\, y} \right)
\end{aligned}
$$

Curl with Levi-Civita:

$$\nabla \times \vec{A} \;=\; \lim_{V \to 0} \frac{\int_S \epsilon_{ijk}\, \hat{x}_i\, d\sigma_j\, A_k}{\int_V d\tau} \qquad (5)$$

with

$$d\sigma_1 = dx_2\, dx_3,\; d\sigma_2 = dx_3\, dx_1,\; d\sigma_3 = dx_1\, dx_2.$$

$$\begin{aligned} & \epsilon_{ijk}\, \hat{x}_i\, d\sigma_j\, A_k \;= \\ & \hat{x}_1\; (d\sigma_2\, \partial_2 A_3\, dx_2 - d\sigma_3\, \partial_3 A_2\, dx_3) \\ +\; & \hat{x}_2\; (d\sigma_3\, \partial_3 A_1\, dx_3 - d\sigma_1\, \partial_1 A_3\, dx_1) \\ +\; & \hat{x}_3\; (d\sigma_1\, \partial_1 A_2\, dx_1 - d\sigma_2\, \partial_2 A_1\, dx_2) \end{aligned}$$

$$\begin{aligned} & \epsilon_{ijk}\, \frac{\hat{x}_i\, d\sigma_j\, A_k}{dx\, dy\, dz} \;= \\ & \hat{x}_1\, (\partial_2 A_3 - \partial_3 A_2) + \hat{x}_2\, (\partial_3 A_1 - \partial_1 A_3) \\ +\; & \hat{x}_3\, (\partial_1 A_2 - \partial_2 A_1) \;=\; \epsilon_{ijk}\, \hat{x}_i\, \partial_j\, A_k\, . \end{aligned}$$