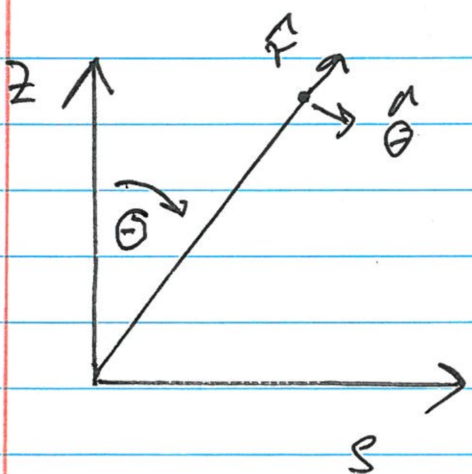
Spherical Coordinates (p.126)

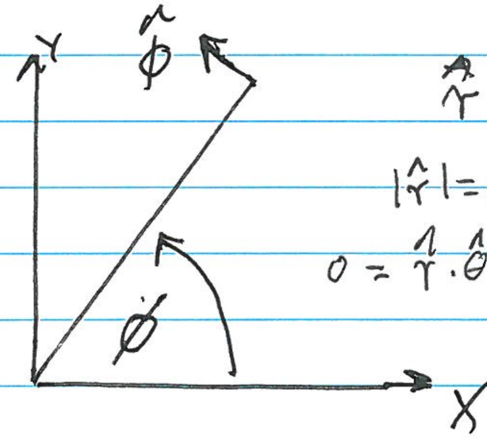
$z = r\cos\theta$, θ polar angle

$\rho = r\sin\theta$ $\swarrow$ ϕ azimuth angle

$y = \rho\sin\phi = r\sin\theta\sin\phi$

$x = \rho\cos\phi = r\sin\theta\cos\phi$





$\hat{r}\times\hat{\theta} = \hat{\phi}$

$|\hat{r}| = |\hat{\theta}| = |\hat{\phi}| = 1$

$0 = \hat{r}\cdot\hat{\theta} = \hat{r}\cdot\hat{\phi}$
$= \hat{\theta}\cdot\hat{\phi}$

$\vec{r} = r\,\hat{r}$

$$d\vec{r} = \hat{r}\,dr + \hat{\theta}\,r\,d\theta + \hat{\phi}\,\underbrace{r\sin\theta}_{\rho}\,d\phi$$

$$\vec{v} = \hat{r}\,\dot{r} + \hat{\theta}\,r\,\dot{\theta} + \hat{\phi}\,r\sin\theta\,\dot{\phi}$$

$$(d\vec{r})^2 = (ds)^2 = (dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta\,(d\phi)^2$$

C10 $\Leftarrow$ $d^3x = (dr)(r\,d\theta)(r\sin\theta\,d\phi)$

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial \phi}$$

Matrices of dot products:

We had already

$$\begin{array}{c|cc} & \hat{x} & \hat{y} \\ \hline \hat{s} & \cos\phi & \sin\phi \\ \hat{\phi} & -\sin\phi & \cos\phi \end{array}$$

Now

$$\begin{array}{c|cc} & \hat{s} & \hat{z} \\ \hline \hat{r} & \sin\theta & \cos\theta \\ \hat{\theta} & \cos\theta & -\sin\theta \end{array}$$

$$\begin{array}{c|ccc} & \hat{x} & \hat{y} & \hat{z} \\ \hline \hat{r} & \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \hat{\theta} & \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ \hat{\phi} & -\sin\phi & \cos\phi & 0 \end{array}$$

Derivatives of spherical unit vectors

$$\frac{\partial}{\partial r}\hat{r} = \frac{\partial}{\partial r}\hat{\theta} = \frac{\partial}{\partial r}\hat{\phi} = 0$$

$$\frac{\partial \hat{r}}{\partial \theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z} = \hat{\theta}$$

$$\frac{\partial \hat{\theta}}{\partial \theta} = -\sin\theta\cos\phi\,\hat{x} - \sin\theta\sin\phi\,\hat{y} - \cos\theta\,\hat{z} = -\hat{r}$$

$$\frac{\partial}{\partial \theta}\hat{\phi} = 0$$

$$\frac{\partial \hat{r}}{\partial \phi} = -\sin\theta\sin\phi\,\hat{x} + \sin\theta\cos\phi\,\hat{y} = \sin\theta\,\hat{\phi}$$

$$\frac{\partial \hat{\theta}}{\partial \phi} = -\cos\theta\sin\phi\,\hat{x} + \cos\theta\cos\phi\,\hat{y} = \cos\theta\,\hat{\phi}$$

$$\frac{\partial \hat{\phi}}{\partial \phi} = -\cos\phi\,\hat{x} - \sin\phi\,\hat{y} = -\rho$$

$$= -\sin\theta\,\hat{r} - \cos\theta\,\hat{\theta}$$

Gradient: $\vec{\nabla}\psi = \hat{r}\frac{\partial \psi}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial \psi}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial \phi}$

Divergence: $\vec{\nabla}\cdot\vec{A} =$

$$\left(\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial \phi}\right)\cdot\left(\hat{r}A_r + \hat{\theta}A_\theta + \hat{\phi}A_\phi\right)$$

$$A_r = \hat{r}\cdot\vec{A},\quad A_\theta = \hat{\theta}\cdot\vec{A},\quad A_\phi = \hat{\phi}\cdot\vec{A}$$

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial A_r}{\partial r} + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$+ \frac{\hat{\theta}}{r}\cdot\frac{\partial \hat{r}}{\partial \theta} A_r + \cancel{\frac{\hat{\theta}}{r}\cdot\frac{\partial \hat{\phi}}{\partial \theta} A_\phi}^{\,0}$$

$$+ \frac{\hat{\phi}}{r\sin\theta}\cdot\left(\frac{\partial \hat{r}}{\partial \phi} A_r + \frac{\partial \hat{\theta}}{\partial \phi} A_\theta + \cancel{\frac{\partial \hat{\phi}}{\partial \theta} A_\phi}^{\,0}\right)$$

$$= \frac{\partial A_r}{\partial r} + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$+ \frac{A_r}{r} + \frac{A_r}{r} + \frac{\cos\theta}{r\sin\theta} A_\theta$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\, A_\theta\right)$$

$$+ \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$ (Book p.129)

$$\vec{\nabla}^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \psi}{\partial \theta}\right)$$

$$+ \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 \psi}{\partial \phi^2}$$

Curl $\vec{\nabla} \times \vec{A} =$

$$\left(\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial \phi}\right) \times \left(\hat{r} A_r + \hat{\theta} A_\theta + \hat{\phi} A_\phi\right) =$$

$$\hat{r} \times \hat{\theta}\frac{\partial A_\theta}{\partial r} + \hat{r} \times \hat{\phi}\frac{\partial A_\phi}{\partial r} +$$

$$\frac{\hat{\theta}}{r} \times \hat{r}\frac{\partial A_r}{\partial \theta} + \frac{\hat{\theta}}{r} \times \hat{\phi}\frac{\partial A_\phi}{\partial \theta} + \cancel{\frac{\hat{\theta}}{r} \times \overset{0}{\frac{\partial \hat{r}}{\partial \theta}}} A_r + \frac{\hat{\theta}}{r} \times \overset{-\hat{r}}{\frac{\partial \hat{\theta}}{\partial \theta}} A_\theta$$

$$\frac{1}{r\sin\theta}\left(\hat{\phi} \times \hat{r}\frac{\partial A_r}{\partial \phi} + \hat{\phi} \times \hat{\theta}\frac{\partial A_\theta}{\partial \phi} + \cancel{\hat{\phi} \times \frac{\partial \hat{r}}{\partial \phi}}_{0} A_r + \cancel{\hat{\phi} \times \frac{\partial \hat{\theta}}{\partial \phi}}_{0} A_\theta + \hat{\phi} \times \frac{\partial \hat{\phi}}{\partial \phi} A_\phi\right)$$

$$= \hat{\phi}\frac{\partial A_\theta}{\partial r} - \hat{\theta}\frac{\partial A_\phi}{\partial r} - \frac{\hat{\phi}}{r}\frac{\partial A_r}{\partial \theta} + \frac{\hat{r}}{r}\frac{\partial A_\phi}{\partial \theta} + \frac{\hat{\phi}}{r} A_\theta +$$

$$\frac{1}{r\sin\theta}\left(\hat{\theta}\frac{\partial A_r}{\partial \phi} - \hat{r}\frac{\partial A_\theta}{\partial \phi} - \hat{\theta}\sin\theta A_\phi + \hat{r}\cos\theta A_\phi\right)$$

$$= \frac{\hat{r}}{r\sin\theta}\left(\frac{\partial}{\partial \theta}(\cos\theta A_\phi) - \frac{\partial}{\partial \phi} A_\theta\right)$$

$$+ \hat{\theta}\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} A_r - \frac{1}{r}\frac{\partial}{\partial r}(r A_\phi)\right)$$

$$+ \frac{\hat{\phi}}{r}\left(\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial}{\partial \theta} A_r\right)$$

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Acceleration: $\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\,\dot{\phi}\,\hat{\phi}$

$$\vec{a} = \dot{\vec{v}} = \ddot{r}\hat{r} + \dot{r}\dot{\hat{r}} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\dot{\hat{\theta}} + \dot{r}\sin\theta\,\dot{\phi}\,\hat{\phi} + r\cos\theta\,\dot{\theta}\dot{\phi}\,\hat{\phi} + r\sin\theta\,\ddot{\phi}\,\hat{\phi} + r\sin\theta\,\dot{\phi}\,\dot{\hat{\phi}}$$

$$\dot{\hat{r}} = \frac{d\hat{r}}{dt} = \frac{\partial\hat{r}}{\partial\theta}\dot{\theta} + \frac{\partial\hat{r}}{\partial\phi}\dot{\phi}$$

$$(\cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z})\,\dot{\theta}$$

$$(-\sin\theta\sin\phi\,\hat{x} + \sin\theta\cos\phi\,\hat{y})\,\dot{\phi}$$

$$= \dot{\theta}\,\hat{\theta} + \sin(\theta)\,\dot{\phi}\,\hat{\phi}$$

$$\dot{\hat{\theta}} = \frac{d\hat{\theta}}{dt} = \frac{\partial\hat{\theta}}{\partial\theta}\dot{\theta} + \frac{\partial\hat{\theta}}{\partial\phi}\dot{\phi} = -\dot{\theta}\,\hat{r} + \cos\theta\,\dot{\phi}\,\hat{\phi}$$

$$\dot{\hat{\phi}} = \frac{d\hat{\phi}}{dt} = -\sin\theta\,\dot{\phi}\,\hat{r} - \cos\theta\,\dot{\phi}\,\hat{\theta}$$

$$\vec{a} = \hat{r}\left(\ddot{r} - r\dot{\theta}^2 - r\sin^2(\theta)\,\dot{\phi}^2\right) +$$
$$\hat{\theta}\left(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin(\theta)\cos(\theta)\,\dot{\phi}^2\right) +$$
$$\hat{\phi}\left(r\sin(\theta)\,\ddot{\phi} + 2\dot{r}\sin\theta\,\dot{\phi} + 2r\cos(\theta)\,\dot{\theta}\,\dot{\phi}\right)$$