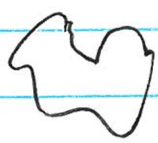
Vector integration ①

(Book p.58)

Line integrals:

$$\int_C \vec{V} \cdot d\vec{r}$$

C ← Contour

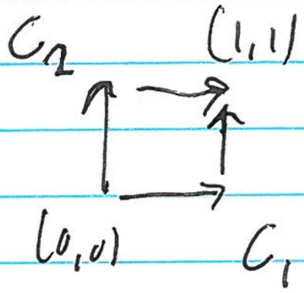


Example: Work

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C F_x \, dx + \int_C F_y \, dy + \int_C F_z \, dz$$

$$d\vec{r} = \hat{x} \, dx + \hat{y} \, dy + \hat{z} \, dz$$

Example 1.9.1 (p.59):



Path-dependent work.

$$\vec{F} = -\hat{x} y + \hat{y} x$$

C_1:
$$W_1 = -\int_0^1 \underbrace{y}_{y=0} \, dx + \int_0^1 \underbrace{x}_{x=1} \, dy = +1$$

C_2:
$$W_2 = +\int_0^1 \underbrace{x}_{x=0} \, dy - \int_0^1 \underbrace{y}_{y=1} \, dx = -1$$

Or
$$W = \oint_{\square} \vec{F} \cdot d\vec{r} = +2.$$

Recall: $\nabla \times \vec{F} = 2\hat{x}_3$

Surface integrals: (p. 62 book)

$$\int_S \vec{V}\cdot d\vec{r} = \int_S \vec{V}\cdot\vec{n}\, dA$$

Surface

Vector function

Normal to surface. $\vec{n}^2 = 1$ unit vector.

Infinitesimal area element

Conventions for normal:

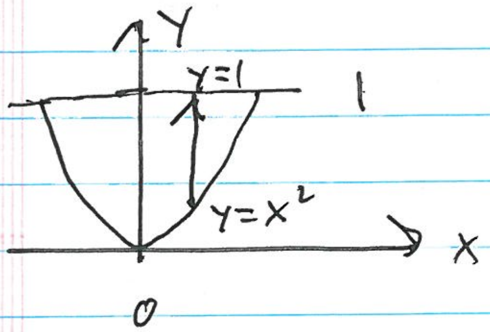
Closed surfaces: $\vec{n}$ points out side.

Open surfaces: Right-hand-rule for direction $\vec{V}$ as $\parallel \hat{z}$.

Example 1.1.4: (book p.63)

Moment of inertia of parabola $y = x^2$.

it is flat, rotate it about the y-axis:



$$I_y = \mu \int_{-1}^{1} dx\, x^2 \int_{y=x^2}^{1} dy \; =$$

mass density (a constant)

$$= 2\mu \int_0^1 dx\, x^2 (1-x^2) = 2\mu \left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Big|_0^1$$

symmetry

$$= 2\mu \left(\frac{1}{3} - \frac{1}{5}\right) = \underline{\underline{\frac{4\mu}{15}}}$$

• Second Solution → Backside

Possible parametrization of a surface in 3D space: $z = f(x,y)$

Example: Saddle point at $\vec{r} = 0$

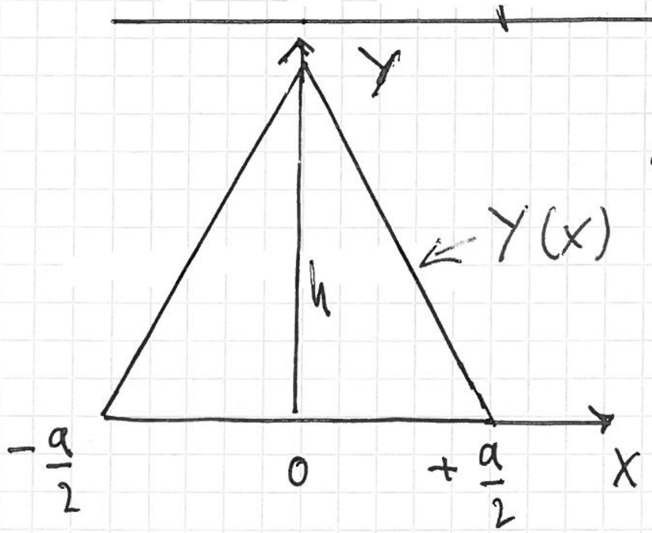
$$z = xy$$

Fig. 1.31 (book p. 64): Drawing on unit circle $x^2+y^2=1$.

(Show)

By integration:

Area of an equilateral triangle:



$$A = 2 \int_0^{a/2} dx \int_0^{h(1-2x/a)}$$

$$y = a_1 x + a_0$$

$x = 0 \Rightarrow y = h \Rightarrow a_0 = h$

$x = \frac{a}{2} \Rightarrow y = 0 = a_1 \frac{a}{2} + h$

$$a_1 = -h\, 2/a$$

$$\left(\frac{a}{2}\right)^2 + h^2 = a^2$$

$h = \frac{\sqrt{3}}{2} a$ height

Vector integration (3)

Backside

$$2\int_0^{\sqrt{y}} x^2\,dy = \frac{2}{3}x^3\Big|_0^{\sqrt{y}} = \frac{2}{3}y^{3/2}$$

$$I_y = \frac{2}{3}\mu\int_0^1 dy\,y^{3/2} = \frac{2}{3}\mu\frac{2}{5}\,y^{\frac{5}{2}}\Big|_0^1$$

$$= \underline{\underline{\frac{4\mu}{15}}}$$

(Moment of line with density 1.)

$$A = 2 \int_0^{a/2} dx \; y(x), \quad y(x) = h\left(1 - \frac{2x}{a}\right)$$

$$= 2h\left(x - \frac{x^2}{a}\right)\Bigg|_0^{a/2} = 2h\left(\frac{a}{2} - \frac{a}{4}\right)$$

$$= 2h\,\frac{a}{4} = \underline{\underline{\frac{ah}{2}}} = \underline{\underline{\frac{\sqrt{3}}{4}\,a^2}}$$

Volume integrals: (book p. 65)

$$\int_V \vec{V}\, d\tau = \int_V V_x\, d\tau + \int_V V_y\, d\tau + \int_V V_z\, d\tau$$

V: Volume

Vector field

Densities

$d\tau = d^3x = d^3r$ are other conventions used.

Volume of a 3D tetrahedron:

Area base triangle: $A = \frac{a_1^0 \, h_1^0}{2}$

New height: h_2^0 (z-direction)

$a_1(z) = bz + a_1^0, \quad 0 = b h_2^0 + a_1^0 \rightsquigarrow b = -\frac{a_1^0}{h_2^0}$

$$a_1(z) = a_1^0 - \frac{a_1^0}{h_2^0} z = a_1^0 \left(1 - z/h_2^0\right)$$

Similarly: $$h_1(z) = h_1^0 \left(1 - z/H_2^0\right)$$

$$V = \frac{1}{2}\int_0^{h_2^0} dz \; a_1(z)\, h_1(z) = \qquad z = h_2^0 z'$$

$$\frac{a_1^0 h_1^0}{2} \int_0^{h_2^0} dz \left(1 - z/h_2^0\right)^2 = \frac{a_1^0 h_1^0}{2} \int_0^1 h_2^0 dz' (1-z')^2$$

$$\boxed{\frac{1}{3}}$$

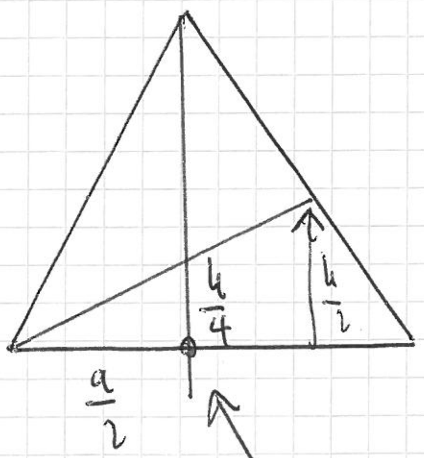
$$V = \frac{a_1^0 h_1^0 h_2^0}{2 \cdot 3}$$

Tetrahedron:

$$h_1^0 = \frac{\sqrt{5}}{2} a_1^0 = \frac{\sqrt{5}}{2} a$$

$$h_2^0 = ?$$

Center of equilateral triangle: $h_1^0 = h$



Starting point for h a.

$$h^2 = \left(\frac{h}{4}\right)^2 + \left(h_2^0\right)^2$$

$$h_1^0 = \frac{\sqrt{15}}{4} h = \frac{5\sqrt{3}}{8} a$$

$$V = \frac{1}{6} \frac{\sqrt{15}}{4} \frac{5\sqrt{3}}{8} a^3$$

$$= \frac{5\,\cancel{\sqrt{15}}}{\cancel{19}2\,64} a^3$$

4 D tetrahedron

$$a_1(w) = bw + a_1^0 \quad \text{etc.}$$
$$= a_1^0(1 - w/h_3^0)$$
$$h_1(w) = h_1^0(1 - w/h_3^0)$$
$$h_2(w) = h_2^0(1 - w/h_3^0)$$

$$V_4 = \int_0^{h_3^0} dw \, \frac{a_1(w) h_1(w) h_2(w)}{2 \cdot 3}$$
$$= \frac{a_1^0 h_1^0 h_2^0}{2 \cdot 3} \int_0^{h_3^0} dw \left(1 - \frac{w}{h_3^0}\right)^3$$
$$= \frac{a_1^0 h_1^0 h_2^0 h_3^0}{2 \cdot 3} \int_0^1 dw' (1-w')^3$$

$$V_4 = \frac{a_1^0 h_1^0 h_2^0 h_3^0}{2 \cdot 3 \cdot 4}$$

nD:
$$V_n = \frac{a_1^0 \prod_{i=1}^{n-1} h_i^0}{n!}$$

Proof by induction.