# Mathematical Physics — PHZ 3113

## Curl Homework (January 30, 2013)

Use the identity (with Einstein convention)

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \qquad (1)$$

to solve the following assignments.

1. Show (Book (1.89) p.51)

$$\vec{B} \times \left(\nabla \times \vec{A}\right) + \vec{A} \times \left(\nabla \times \vec{B}\right) =$$
$$\nabla \left(\vec{A} \cdot \vec{B}\right) - \left(\vec{B} \cdot \nabla\right) \vec{A} - \left(\vec{A} \cdot \nabla\right) \vec{B} \ . \qquad (2)$$

Solution ($\partial_i = \frac{\partial}{\partial x_i}$ in the following):

$$\begin{aligned}
&\vec{B} \times \left(\nabla \times \vec{A}\right) + \vec{A} \times \left(\nabla \times \vec{B}\right) \\
=\ &\epsilon_{ijk}\epsilon_{klm}\hat{x}_i B_j \partial_l A_m + \epsilon_{ijk}\epsilon_{klm}\hat{x}_i A_j \partial_l B_m \\
=\ &\left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right) \hat{x}_i B_j \partial_l A_m \ + \\
&\left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right) \hat{x}_i A_j \partial_l B_m \\
=\ &\hat{x}_i B_j \partial_i A_j - \hat{x}_i B_j \partial_j A_i \ + \\
&\hat{x}_i A_j \partial_i B_j - \hat{x}_i A_j \partial_j B_i \\
=\ &\hat{x}_i \partial_i \left(B_j A_j\right) - \left(\vec{B} \cdot \nabla\right) \vec{A} - \left(\vec{A} \cdot \nabla\right) \vec{B} \\
=\ &\nabla \left(\vec{A} \cdot \vec{B}\right) - \left(\vec{B} \cdot \nabla\right) \vec{A} - \left(\vec{A} \cdot \nabla\right) \vec{B} \ .
\end{aligned}$$

2. The vector potential $\vec{A}$ of a magnetic dipole moment $\vec{m}$ is given by (Book 1.7.11 p.52)

$$\vec{A} = \left(\frac{\mu_0}{4\pi}\right)\left(\frac{\vec{m}\times\vec{r}}{r^3}\right).$$

Calculate the magnetic field $\vec{B} = \nabla\times\vec{A}$. Solution ($\partial_i = \frac{\partial}{\partial x_i}$ in the following and we shall use $\partial_i f(r) = \frac{df}{dr}\frac{x_i}{r}$):

$$\begin{aligned}
&\left(\frac{4\pi}{\mu_0}\right)\nabla\times\vec{A} = \nabla\times\left(\frac{\vec{m}\times\vec{r}}{r^3}\right)\\
&= \epsilon_{ijk}\epsilon_{klm}\hat{x}_i\partial_j\frac{m_l x_m}{r^3}\\
&= \epsilon_{ijk}\epsilon_{klm}\hat{x}_i\frac{m_l\delta_{jm}}{r^3} - 3\,\epsilon_{ijk}\epsilon_{klm}\hat{x}_i\frac{m_l x_j x_m}{r^5}\\
&= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\hat{x}_i\frac{m_l\delta_{jm}}{r^3} -\\
&\quad 3\left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\hat{x}_i\frac{m_l x_j x_m}{r^5}\\
&= \frac{3\vec{m}}{r^3} - \frac{\vec{m}}{r^3} - \frac{3\vec{m}\,r^2}{r^5} + \frac{3\vec{r}\,(\vec{m}\cdot\vec{r})}{r^5}\\
&= \frac{3\hat{r}\,(\vec{m}\cdot\hat{r}) - \vec{m}}{r^3}.
\end{aligned}$$

$$\vec{B} \;=\; \left(\frac{\mu_0}{4\pi}\right) \frac{3\,\hat{r}\;(\vec{m}\cdot\hat{r}) - \vec{m}}{r^3}\,. \tag{3}$$

Second solution:

$$\nabla \times \left(\frac{\vec{m}\times\vec{r}}{r^3}\right) \;=\; \frac{\vec{r}}{r^3}\cdot\nabla\,\vec{m} \;+\; \vec{m}\left(\nabla\cdot\frac{\vec{r}}{r^3}\right)$$

$$-\;\vec{m}\cdot\nabla\,\frac{\vec{r}}{r^3} \;-\; \frac{\vec{r}}{r^3}\,\nabla\cdot\vec{m}$$

$$=\;\vec{m}\left(\nabla\cdot\frac{\vec{r}}{r^3}\right) \;-\; \vec{m}\cdot\nabla\,\frac{\vec{r}}{r^3}$$

because the derivatives acting on the constant vector $\vec{m}$ are zero. Now,

$$\left(\nabla\cdot\frac{\vec{r}}{r^3}\right) \;=\; \frac{3}{r^3} - \vec{r}\cdot\hat{r}\frac{3}{r^4} \;=\; 0\,,$$

$$-\;\vec{m}\cdot\nabla\,\frac{\vec{r}}{r^3} \;=\; -\;\frac{1}{r^3}\sum_{i=1}^{3} m_i\partial_i \sum_{j=1}^{3}\hat{x}_j x_j$$

$$-\;\vec{r}\sum_{i=1}^{3} m_i\partial_i\frac{1}{r^3} \;=$$

$$-\;\frac{1}{r^3}\sum_{i=1}^{3}\sum_{j=1}^{3} m_i\hat{x}_j\delta_{ij} \;+\; \vec{r}\;(\vec{m}\cdot\vec{r})\;\frac{3}{r^5} \;=$$

$$-\;\frac{\vec{m}}{r^3} \;+\; \frac{3\,\vec{r}\;(\vec{m}\cdot\vec{r})}{r^5} \;=\; \frac{3\,\hat{r}\;(\vec{m}\cdot\hat{r}) - \vec{m}}{r^3}$$

and $\vec{B}$ as before.