Mathematical Physics — PHZ 3113
Curl; Vector Integration Homework
(February 7, 2013)

The dual of the Euclidean electromagnetic field tensor is defined by

$$^*F_{\alpha\beta} = \frac{1}{2}\,\epsilon_{\alpha\beta\mu\nu}F_{\mu\nu} \tag{1}$$

where the indices run from 1 to 4, the Einstein convention is used and $F_{\mu\nu}$ is an anti-symmetric (i.e., $F_{\mu\nu} = -F_{\nu\mu}$) rank two tensor

$$(F_{\mu\nu}) \;=\; \begin{pmatrix} 0 & F_{12} & F_{13} & F_{14} \\ -F_{12} & 0 & F_{23} & F_{24} \\ -F_{13} & -F_{23} & 0 & F_{34} \\ -F_{14} & -F_{24} & -F_{34} & 0 \end{pmatrix}.$$

1. Calculate the elements of $^*F_{\alpha\beta}$ in terms of the independent elements of $F_{\mu\nu}$ and write $^*F_{\alpha\beta}$ as a matrix.

   Solution:

$$^*F_{12} \;=\; \frac{1}{2}(\epsilon_{1234}F_{34} + \epsilon_{1243}F_{43}) \;=\; +F_{34}$$

$$
\begin{aligned}
{}^*F_{13} &= \frac{1}{2}(\epsilon_{1324}F_{24} + \epsilon_{1342}F_{42}) &= -F_{24} \\
{}^*F_{14} &= \frac{1}{2}(\epsilon_{1423}F_{23} + \epsilon_{1432}F_{32}) &= +F_{23} \\
{}^*F_{23} &= \frac{1}{2}(\epsilon_{2314}F_{14} + \epsilon_{2341}F_{41}) &= +F_{14} \\
{}^*F_{24} &= \frac{1}{2}(\epsilon_{2413}F_{13} + \epsilon_{2431}F_{31}) &= -F_{13} \\
{}^*F_{34} &= \frac{1}{2}(\epsilon_{3412}F_{12} + \epsilon_{3421}F_{21}) &= +F_{12}
\end{aligned}
$$

$$
\left({}^*F_{\alpha\beta}\right) = \begin{pmatrix} 0 & +F_{34} & -F_{24} & +F_{23} \\ -F_{34} & 0 & +F_{14} & -F_{13} \\ +F_{24} & -F_{14} & 0 & +F_{12} \\ -F_{23} & +F_{13} & -F_{12} & 0 \end{pmatrix} .
$$

2. The electromagnetic field tensor can be written in form of derivatives of a 4-potential

$$
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \, . \tag{2}
$$

Use this expression to proof the homogeneous Maxwell equations in their form

$$
\partial_\alpha \, {}^*F_{\alpha\beta} = 0 \, . \tag{3}
$$

Solution:

$$\begin{aligned}\partial_\alpha\ {}^*F_{\alpha\beta} &= \frac{1}{2}\,\epsilon_{\alpha\beta\mu\nu}\,\partial_\alpha\,F_{\mu\nu}\\ &= \frac{1}{2}\,\epsilon_{\alpha\beta\mu\nu}\,\partial_\alpha\,\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\end{aligned}$$

Now

$$\begin{aligned}\epsilon_{\alpha\beta\mu\nu}\,\partial_\alpha\,\partial_\mu\,A_\nu &= 0\\ \epsilon_{\alpha\beta\mu\nu}\,\partial_\alpha\,\partial_\nu\,A_\mu &= 0\end{aligned}$$

because each time an antisymmetric tensor is contracted with a symmetric tensor ($\partial_\alpha\,\partial_\mu$ and $\partial_\alpha\,\partial_\mu$). This proofs (3).

Remark: Therefore (2) implies that no magnetic monopoles exist, because (3) includes (upon identification of the $\vec{E}$ and $\vec{B}$ fields) the relation $\nabla\cdot\vec{B}=0$.