1. Calculate

$$I = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx\, dy\, e^{-(x+y)^2-(x-y)^2} .$$

Solution:

$$\xi = x + y , \qquad \eta = x - y ,$$
$$x = \frac{1}{2}(\xi + \eta) , \qquad y = \frac{1}{2}(\xi - \eta) ,$$
$$\begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2} .$$

The minus comes because the transformation switches the righ-handed system

$$d\vec{x} = \hat{x}\, dx , \quad d\vec{y} = \hat{y}\, dy$$

into the left-handed system

$$d\vec{\xi} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} d\xi , \quad d\vec{\eta} = \begin{pmatrix} 0.5 \\ -0.5 \end{pmatrix} d\eta .$$

Interchanging the definitions of ξ and η to $\xi = x - y$ and $\eta = x + y$ gives a right-handed system and Jacobian $+1/2$. Therefore,

$$I = \frac{1}{2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\xi\, d\eta\, e^{-\xi^2-\eta^2} = \frac{\pi}{2} .$$

2. Calculate

$$I = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx_1\, dx_2\, dx_3 \; e^{-(x_1+x_2+x_3)^2-(x_1+x_2-x_3)^2-(x_1-x_2-x_3)^2} .$$

Solution:

$$\begin{aligned} \xi_1 &= x_1 + x_2 + x_3 , \\ \xi_2 &= x_1 + x_2 - x_3 , \\ \xi_3 &= x_1 - x_2 - x_3 , \end{aligned}$$

$$\begin{aligned} x_1 &= \frac{1}{2}\left(\xi_1 + \xi_3\right) , \\ x_2 &= \frac{1}{2}\left(\xi_2 - \xi_3\right) , \\ x_3 &= \frac{1}{2}\left(\xi_1 - \xi_2\right) . \end{aligned}$$

Therefore,

$$\begin{vmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} & \frac{\partial x_1}{\partial \xi_3} \\ \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} & \frac{\partial x_2}{\partial \xi_3} \\ \frac{\partial x_3}{\partial \xi_1} & \frac{\partial x_3}{\partial \xi_2} & \frac{\partial x_3}{\partial \xi_3} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{vmatrix} = -\frac{1}{4} .$$

Again the minus sign is due to a switch from a right-handed to a left-handed system and the integral becomes

$$I = \frac{1}{4} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\xi_1 \, d\xi_2 \, d\xi_3$$

$$e^{-(\xi_1)^2-(\xi_2)^2-(\xi_3)^2} = \frac{\pi^{3/2}}{4} \, .$$