Mathematical Physics — PHZ 3113

Levi-Cevita Tensor 1

(January 11, 2013)

Group #

Participating students (print):

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1. Use binary numbers 0, 1 and write down the numbers 0 to 3. Add one more column in which you substitute $0 \to 1$, $1 \to 2$ and one last column in which you count the decimal numbers from 1 to 4.

   0 00 11 1
   1 01 12 2
   2 10 21 3
   3 11 22 4

2. Use numbers with base 3 and symbols 0, 1, 2 to write down the numbers 0 to 26. Add one more column in which you

substitute $n \to n+1$ for $n = 0,\ 1,\ 2$ and one last column in which you count the decimal numbers from 1 to 27.

0	000	111	1
1	001	112	2
2	002	113	3
3	010	121	4
4	011	122	5
5	012	123	6
6	020	131	7
7	021	132	8
8	022	133	9
9	100	211	10
10	101	212	11
11	102	213	12
12	110	221	13
13	111	222	14
14	112	223	15
15	120	231	16
16	121	232	17
17	122	233	18

18 200 311 19
19 201 312 20
20 202 313 21
21 210 321 22
22 211 322 23
23 212 323 24
24 220 331 25
25 221 332 26
26 222 333 27

3. Write down the **permutations** of 1 2. How many permutations of 1 2 are there?

$$1\,2 \quad 2\,1 \quad \text{there are 2 permuations.} \tag{1}$$

4. Write down the transposition of 1 2.

$$2\,1 \tag{2}$$

5. Use the permutations of (1) to write down the permutations of 1 2 3 by starting in each case with 3 on the right and transposing the number 3 with its left neighbor until this is no longer possible. How

many permutations of 1 2 3 are there?

$$\begin{matrix} 123 & 132 & 312 \\ 213 & 231 & 321 \end{matrix} \qquad (3)$$

There are 6 permutations.

6. Along the same lines: Use the permutations of (3) to write down the permutations of 1 2 3 4. How many permutations of 1 2 3 4 are there?

$$\begin{matrix} 1234 & 1243 & 1423 & 4123 \\ 1324 & 1342 & 1432 & 4132 \\ 3124 & 3142 & 3412 & 4312 \\ 2134 & 2143 & 2413 & 4213 \\ 2314 & 2341 & 2431 & 4231 \\ 3214 & 3241 & 3421 & 4321 \end{matrix} \qquad (4)$$

There are $24 = 4!$ permutations.

7. Proof that there are $n!$ permutions $\pi_1, \ldots, \pi_n$ of $1, \ldots, n$.

Thee proof is by induction. Assume that

there are $(n-1)!$ permutations for

$$\pi_1, \ldots, \pi_{n-1}$$

as we have verified up to $n-1 = 4$. From each of these permuations we get n new permuations by starting with the number n on the right side and transposing it in $(n-1)$ steps to the left side:

$$\begin{array}{c} \pi_1, \ \ldots \ , \pi_{n-1}, n \\ \pi_1, \ \ldots \ , n, \pi_{n-1} \\ \ldots \\ \pi_1, n, \ \ldots \ , \pi_{n-1} \\ n, \pi_1, \ \ldots \ , \pi_{n-1} \end{array}$$

Hence there are

$$(n-1)!\, n \ = \ n!$$

permuations of the numbers $1, \ldots, n$.

8. With

$$\begin{aligned} i_1 &= 1, \ldots, n\,, \\ i_2 &= 1, \ldots, n\,, \\ &\ldots \\ i_n &= 1, \ldots, n \end{aligned}$$

the definition of the **Levi-Cevita tensor** is

$$\epsilon_{i_1,\ldots,i_n} = \begin{cases} +1 \text{ for } i_1, \ldots, i_n \text{ even permutation,} \\ -1 \text{ for } i_1, \ldots, i_n \text{ odd permutation,} \\ 0 \text{ for } i_1, \ldots, i_n \text{ no permutation.} \end{cases} \tag{5}$$

A permutation $i_1, \ldots, i_n$ is even, when it is generated by an even number of transpositions of $1, \ldots, n$ and odd, when it is generated by an odd number of transpositions of $1, \ldots, n$.

How many non-zero elements are there?

$$n! \tag{6}$$

How many positive elements are there?

$$n!/2 \tag{7}$$

9. Write down all elements of the Levi-Cevita tensor for $n = 2$.

$$\begin{aligned} \epsilon_{11} &= 0 \quad \epsilon_{12} = +1 \\ \epsilon_{21} &= -1 \quad \epsilon_{22} = 0 \end{aligned} \tag{8}$$

10. Write down all elements of the Levi-Cevita tensor for $n = 3$.

$$\begin{aligned}
1.\ \epsilon_{111} &= 0 \\
2.\ \epsilon_{112} &= 0 \\
3.\ \epsilon_{113} &= 0 \\
4.\ \epsilon_{121} &= 0 \\
5.\ \epsilon_{122} &= 0 \\
6.\ \epsilon_{123} &= +1 \\
7.\ \epsilon_{131} &= 0 \\
8.\ \epsilon_{132} &= -1 \\
9.\ \epsilon_{133} &= 0 \\
10.\ \epsilon_{211} &= 0 \\
11.\ \epsilon_{212} &= 0 \\
12.\ \epsilon_{213} &= -1 \\
13.\ \epsilon_{221} &= 0 \\
14.\ \epsilon_{222} &= 0
\end{aligned} \tag{9}$$

15. $\epsilon_{223} = 0$
16. $\epsilon_{231} = +1$
17. $\epsilon_{232} = 0$
18. $\epsilon_{233} = 0$
19. $\epsilon_{311} = 0$
20. $\epsilon_{312} = +1$
21. $\epsilon_{313} = 0$
22. $\epsilon_{321} = -1$
23. $\epsilon_{322} = 0$
24. $\epsilon_{323} = 0$
25. $\epsilon_{331} = 0$
26. $\epsilon_{332} = 0$
27. $\epsilon_{333} = 0$

11. Write down all non-zero elements of the Levi-Cevita tensor for $n = 4$.

    1. $\epsilon_{1234} = +1$
    2. $\epsilon_{1243} = -1$
    3. $\epsilon_{1423} = +1$
    4. $\epsilon_{4123} = -1$

5. $\epsilon_{1324} = -1$
6. $\epsilon_{1342} = +1$
7. $\epsilon_{1432} = -1$
8. $\epsilon_{4132} = +1$

9. $\epsilon_{3124} = +1$
10. $\epsilon_{3142} = -1$
11. $\epsilon_{3412} = +1$
12. $\epsilon_{4312} = -1$

13. $\epsilon_{2134} = -1$
14. $\epsilon_{2143} = +1$
15. $\epsilon_{2413} = -1$
16. $\epsilon_{4213} = +1$

17. $\epsilon_{2314} = +1$
18. $\epsilon_{2341} = -1$
19. $\epsilon_{2431} = +1$
20. $\epsilon_{4213} = -1$

21. $\epsilon_{3214} = -1$
22. $\epsilon_{3241} = +1$
23. $\epsilon_{3241} = -1$
24. $\epsilon_{4321} = +1$