Mathematical Physics — PHZ 3113

Levi-Cevita Tensor Homework 1

(January 23, 2013)

1. Use the $3D$ identity

$$\sum_{i=1}^{3} \epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \quad (1)$$

to calculate

$$(\hat{a} \times \hat{b}) \cdot (\hat{a} \times \hat{b}) \quad (2)$$

where $\hat{a}$ and $\hat{b}$ are unit vectors. Eliminate all $\hat{a} \cdot \hat{b}$ in favor of $\cos\theta$.

Solution: Let

$$\hat{a} = \begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \\ \tilde{a}_3 \end{pmatrix} \text{ and } \hat{b} = \begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \tilde{b}_3 \end{pmatrix}. \quad (3)$$

Then

$$(\hat{a} \times \hat{b}) \cdot (\hat{a} \times \hat{b}) = \quad (4)$$

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \epsilon_{ijk}\epsilon_{ilm}\tilde{a}_j\tilde{b}_k\tilde{a}_l\tilde{b}_m =$$

$$\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \left(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\right) \tilde{a}_j\tilde{b}_k\tilde{a}_l\tilde{b}_m =$$

$$\sum_{l=1}^{3}\sum_{m=1}^{3}\left(\tilde{a}_l\tilde{b}_m\tilde{a}_l\tilde{b}_m - \tilde{a}_m\tilde{b}_l\tilde{a}_l\tilde{b}_m\right) =$$
$$(\hat{a}\cdot\hat{a})\left(\hat{b}\cdot\hat{b}\right) - \left(\hat{a}\cdot\hat{b}\right)\left(\hat{b}\cdot\hat{a}\right) =$$
$$1-\cos^2(\theta) = \sin^2(\theta)\,.$$

2. Show $\vec{a}\times\vec{b} = a\,b\,\sin\theta$ (book p.21).

From (3) we have $\hat{a}\times\hat{b} = \sin\theta$. Hence,

$$\vec{a}\times\vec{b} = a\,b\,\hat{a}\times\hat{b} = a\,b\,\sin\theta\,. \quad (5)$$