Mathematical Physics — PHZ 3113

Classwork 13 (April 10, 2013)

Solutions Linear Equations

After finding solutions: Check that they are correct!

1. Find all solutions of the equations

$$x + 2y = 3\,, \tag{1}$$

$$3x + y = 2\,. \tag{2}$$

Solution:

$$\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -5\,, \quad \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} = -1\,, \quad \begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} = -7$$

and by Cramer's rule

$$x = \frac{1}{5}\,, \quad y = \frac{7}{5}\,.$$

Check: Insert x and y into (1) and (2). Alternative way to the solution: Gauss elimination.

2. Find all solutions of the equations

$$x + 2y = 3\,, \tag{3}$$
$$2x + 4y = 6\,. \tag{4}$$

Solution:

$$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0\,, \quad \begin{vmatrix} 3 & 2 \\ 6 & 4 \end{vmatrix} = 0\,, \quad \begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 0$$

and the two equations are linearly dependent. Dividing (4) by 2 gives (3) again. Therefore the solution is

$$y = \frac{3}{2} - \frac{x}{2}$$

for general x.

3. Find all solutions of the equations

$$x + 2y = 3\,, \tag{5}$$
$$2x + 4y = 4\,. \tag{6}$$

Solution:

$$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0\,, \quad \begin{vmatrix} 3 & 2 \\ 4 & 4 \end{vmatrix} = 4\,, \quad \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = -2$$

and there are no solutions. The equations contradict one another. Subtracting (5) from (6) gives

$$x + 2y = 1 , \tag{7}$$

which disagrees with (5).

4. Find all solutions of the equations

$$x + 2y = 0 , \tag{8}$$

$$3x + y = 0 . \tag{9}$$

Solution: As the determinant

$$\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -5$$

is non-zero there is only the trivial solution $x = y = 0$.

5. Find all solutions of the equations

$$x + 2y = 0 , \tag{10}$$

$$2x + 4y = 0 . \tag{11}$$

Solution: From the deterninant

$$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} \;=\; 0\,,$$

we read off

$$\frac{x}{y} \;=\; \frac{4}{-2} \;=\; -2$$

which, of course, follows also directly from either (10) or (11).

6. Find all solutions of the equations

$$x + 2y + z \;=\; 0\,, \qquad (12)$$

$$2x + 4y + z \;=\; 0\,, \qquad (13)$$

$$x + y + 2z \;=\; 0\,. \qquad (14)$$

Solution: As the determinant

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 1 & 1 & 2 \end{vmatrix} \;=\; 8+2+2-1-8-4 \;=\; -1$$

is non-zero there is only the trivial solution $x = y = z = 0$.

7. Find all solutions of the equations

$$x + 2y + z = 0\,, \qquad (15)$$

$$2x + 4y + 2z = 0\,, \qquad (16)$$

$$x + y + 2z = 0\,. \qquad (17)$$

Solution: From the determinant

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 2 \end{vmatrix} = 8 + 4 + 2 - 2 - 8 - 4 = 0$$

we read off

$$\frac{x}{y} = -\frac{\begin{vmatrix} 4 & 2 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix}} = -\frac{6}{2} = -3\,, \qquad (18)$$

$$\frac{x}{z} = +\frac{\begin{vmatrix} 4 & 2 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix}} = -\frac{6}{-2} = -3\,, \qquad (19)$$

$$\frac{y}{z} = -\frac{\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix}} = -\frac{2}{-2} = +1\,. \tag{20}$$

Multiplying (18) with (20) we find consistency with (19). Choosing the normalization $y = 1$ a solution is

$$x = -3\,, \quad y = 1\,, \quad z = 1\,.$$

Inserting these numbers into equations (15), (16) and (17), we check that the solution is correct.