Mathematical Physics — PHZ 3113

Homework 10 (April 10, 2013)

Pauli Matrices

1. Find three 2×2 matrices σ_i, $i = 1, 2, 3$, which fulfill the relations

$$\sigma_i \, \sigma_j = i \, \epsilon_{ijk} \, \sigma_k \quad \text{for} \quad i \neq j \, , \quad (1)$$
$$\sigma_i \, \sigma_j + \sigma_j \, \sigma_i = 2 \, \delta_{ij} \, 1_2 \, , \quad (2)$$

where 1_2 is the 2×2 unit matrix. Solution:

Simple matrices are found from the ansatz

$$\sigma = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad \text{or} \quad \sigma = \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix} \, .$$

From (2) $(\sigma_i)^2 = 1_2$ follows for $i = 1, 2, 3$ and therefore

$$a^2 = b^2 = 1 \quad \text{and} \quad c \, d = 1$$

with solutions

$$a = \pm 1, \; b = \pm 1, \; c = d = \pm 1, \; c = -d = \pm i \, .$$

Due to (1) the unit matrix is excluded as a solution for σ_i. With the choice $c = 1$,

$d = -1$ for σ_1 and $c = -i$, $d = i$ for for σ_2 we have

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad (3)$$

and find σ_3 from (1):

$$\sigma_3 = -i \, \sigma_1 \, \sigma_2 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \qquad (4)$$

Relation (1) follows by explicit multiplication of the matrices and for $i \neq j$ relation (2) is a consequence of the antisymmetry of (1) in the indices i and j.

2. Find the transformation, which generates the general solution σ_i', $i = 1, 2, 3$ from your previous special solution.

Solution: Let A be a 2×2 matrix with

$$|A| = \det(A) \neq 0\,.$$

Then

$$\sigma_i' = A^{-1}\,\sigma_i\,A \qquad (5)$$

is the general solution to Eq. (1) and (2).