Solution Paul Trap

In the quasi-static approximation the field is the electrostatic field with the given boundary conditions at the time in question. Due to the cylindrical symmetry we have $\Phi = \Phi(\rho, z; t)$. To get the boundary conditions on the end electrodes right, we set

$$\Phi \;=\; \left(z^2 - \frac{1}{2}\,\rho^2 - d^2\right)\, F(\rho, z, t)\,.$$

The boundary condition on the ring electrode implies $\rho^2/2 = z^2 + d^2/2$ and on this boundary

$$\Phi \;=\; V_0\,\sin(\omega t) \;=\; -\frac{3}{2}\,d^2\,F(\rho, z, t)\,.$$

Therefore, the function F depends only on t

$$F(t) \;=\; -\frac{2}{3}\,\frac{V_0}{d^2}\,\sin(\omega t)\,,$$

so that the potential becomes

$$\Phi \;=\; -\frac{2}{3}\,\frac{V_0}{d^2}\,\left(z^2 - \frac{1}{2}\,\rho^2 - d^2\right)\,\sin(\omega t)\,,$$

which is now easily seen to be the solution:

$$\nabla^2\Phi = -\frac{2}{3}\frac{V_0}{d^2}\,\sin(\omega t)\,\nabla^2\left(z^2 - \frac{1}{2}\,\rho^2 - d^2\right) \;=\; 0\,.$$

The electric field is given by

$$\vec{E} \;= -\nabla\cdot\Phi \;=\; -\left(\hat{\rho}\,\frac{\partial\Phi}{\partial\rho} + \hat{z}\,\frac{\partial\Phi}{\partial z}\right) \;=\; \frac{2}{3}\,\frac{V_0}{d^2}\,\left(2\,z\,\hat{z} - \rho\,\hat{\rho}\right)\,\sin(\omega t)\,.$$