PHZ3113: Solution for Homework 11.

1. For small oscillation we have derived the Euler-Lagrange equations, which read in matrix notation

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \ddot{\phi} \\ \ddot{\psi} \end{pmatrix} + \begin{pmatrix} 2g/l & 0 \\ 0 & g/l \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = 0 \ .$$

This is solved by the exponential ansatz (physical is the real part of the solution):

$$\Phi(t) = \begin{pmatrix} \phi \\ \psi \end{pmatrix} = e^{i\omega t} \begin{pmatrix} \phi_0 \\ \psi_0 \end{pmatrix} \Rightarrow \begin{pmatrix} \ddot{\phi} \\ \ddot{\psi} \end{pmatrix} = -\omega^2 e^{i\omega t} \begin{pmatrix} \phi_0 \\ \psi_0 \end{pmatrix}$$

$$\begin{pmatrix} -2\omega^2 + 2g/l & -\omega^2 \\ -\omega^2 & -\omega^2 + g/l \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = 0 \ .$$

$$0 = \det \begin{vmatrix} -2\omega^2 + 2g/l & -\omega^2 \\ -\omega^2 & -\omega^2 + g/l \end{vmatrix} = \omega^4 - \frac{4g}{l}\omega^2 + \frac{2g^2}{l^2}$$

with eigenmodes

$$\omega_\pm = \sqrt{\frac{g}{l}} \sqrt{2 \pm \sqrt{2}} \ .$$

2. For the ω_+ eigenvalue we have

$$0 = \frac{g}{l} \left(\begin{pmatrix} -2 - 2\sqrt{2} \end{pmatrix} \begin{pmatrix} -2 - \sqrt{2} \end{pmatrix} \\ \begin{pmatrix} -2 - \sqrt{2} \end{pmatrix} \begin{pmatrix} -1 - \sqrt{2} \end{pmatrix} \right) \begin{pmatrix} \phi_+ \\ \psi_+ \end{pmatrix} \Rightarrow \frac{\psi_+}{\phi_+} = \frac{-2 - 2\sqrt{2}}{2 + \sqrt{2}} = -\sqrt{2}$$

and for ω_-

$$0 = \frac{g}{l} \left(\begin{pmatrix} -2 + 2\sqrt{2} \end{pmatrix} \begin{pmatrix} -2 + \sqrt{2} \end{pmatrix} \\ \begin{pmatrix} -2 + \sqrt{2} \end{pmatrix} \begin{pmatrix} -1 + \sqrt{2} \end{pmatrix} \right) \begin{pmatrix} \phi_+ \\ \psi_+ \end{pmatrix} \Rightarrow \frac{\psi_+}{\phi_+} = \frac{-2 + 2\sqrt{2}}{2 - \sqrt{2}} = \sqrt{2} \ .$$

Therefore, the solutions (real part) can be written

$$\begin{aligned} \phi(t) &= \phi_+(t) + \phi_-(t) \ , \\ \psi(t) &= -\sqrt{2}\,\phi_+(t) + \sqrt{2}\,\phi_-(t) \ . \end{aligned}$$

with

$$\begin{aligned} \phi_+(t) &= A_+ \cos(\omega_+ t) + B_+ \sin(\omega_+ t) \ , \\ \phi_-(t) &= A_- \cos(\omega_- t) + B_- \sin(\omega_- t) \ , \end{aligned}$$

3. The four constants are determined by the four initial value, e.g., ϕ_0, $\dot{\phi}_0$, ψ_0, $\dot{\psi}_0$ at time $t = 0$:

$$\begin{aligned} \phi_0 &= A_+ + A_- \ , & \psi_0 &= \sqrt{2}\,(-A_+ + A_-) \ , \\ \dot{\phi}_0 &= \omega_+ B_+ + \omega_- B_- \ , & \dot{\psi}_0 &= \sqrt{2}\,(-\omega_+ B_+ + \omega_- B_-) \ , \end{aligned}$$

which gives

$$\begin{aligned} A_+ &= \frac{\phi_0}{2} - \frac{\psi_0}{2\sqrt{2}} \ , & A_- &= \frac{\phi_0}{2} + \frac{\psi_0}{2\sqrt{2}} \ , \\ B_+ &= \frac{\dot{\phi}_0}{2\omega_+} - \frac{\dot{\psi}_0}{2\sqrt{2}\,\omega_+} \ , & B_- &= \frac{\dot{\phi}_0}{2\omega_-} + \frac{\dot{\psi}_0}{2\sqrt{2}\,\omega_-} \ . \end{aligned}$$