PHZ3113: Solution for Homework 11.

1. For small oscillation we have derived the Euler-Lagrange equations, which read in matrix notation

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \ddot\phi \\ \ddot\psi \end{pmatrix} + \begin{pmatrix} 2g/l & 0 \\ 0 & g/l \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = 0 \,.$$

This is solved by the exponential ansatz (physical is the real part of the solution):

$$\Phi(t) = \begin{pmatrix} \phi \\ \psi \end{pmatrix} = e^{i\,\omega\,t} \begin{pmatrix} \phi_0 \\ \psi_0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} \ddot\phi \\ \ddot\psi \end{pmatrix} = -\omega^2\, e^{i\,\omega\,t} \begin{pmatrix} \phi_0 \\ \psi_0 \end{pmatrix}$$

$$\begin{pmatrix} -2\omega^2 + 2g/l & -\omega^2 \\ -\omega^2 & -\omega^2 + g/l \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = 0 \,.$$

$$0 = \det \begin{vmatrix} -2\omega^2 + 2g/l & -\omega^2 \\ -\omega^2 & -\omega^2 + g/l \end{vmatrix} = \omega^4 - \frac{4g}{l}\,\omega^2 + \frac{2g^2}{l^2}$$

with eigenmodes

$$\omega_\pm = \sqrt{\frac{g}{l}}\,\sqrt{2 \pm \sqrt{2}} \,.$$

2. Let us take minors with respect to the first row of the determinant. For the ω_+ frequency the ratio of the two minors is

$$\frac{\triangle_{1+}}{\triangle_{2+}} = \frac{-1-\sqrt{2}}{2+\sqrt{2}} = \frac{(-1-\sqrt{2})\,(1-\sqrt{2})}{(2+\sqrt{2})\,(1-\sqrt{2})} = \frac{1}{-\sqrt{2}}$$

and for ω_- it is

$$\frac{\triangle_{1-}}{\triangle_{2-}} = \frac{-1+\sqrt{2}}{2-\sqrt{2}} = \frac{(-1+\sqrt{2})\,(1+\sqrt{2})}{(2-\sqrt{2})\,(1+\sqrt{2})} = \frac{1}{\sqrt{2}} \,.$$

Therefore, the solutions (real part) can be written

$$\begin{aligned} \phi_+(t) &= A_+ \cos(\omega_+ t) + B_+ \sin(\omega_+ t) \,, \\ \phi_-(t) &= A_- \cos(\omega_- t) + B_- \sin(\omega_- t) \,, \\ \phi(t) &= \phi_+(t) + \phi_-(t) \,, \\ \psi(t) &= -\sqrt{2}\,\phi_+(t) + \sqrt{2}\,\phi_-(t) \,. \end{aligned}$$

3. The four constants are determined by the four initial value, e.g., ϕ_0, $\dot\phi_0$, ψ_0, $\dot\psi_0$ at time $t = 0$:

$$\begin{aligned} \phi_0 &= A_+ + A_- \,, & \psi_0 &= \sqrt{2}\,(-A_+ + A_-) \,, \\ \dot\phi_0 &= \omega_+ B_+ + \omega_- B_- \,, & \dot\psi_0 &= \sqrt{2}\,(-\omega_+ B_+ + \omega_- B_-) \,, \end{aligned}$$

which gives

$$\begin{aligned} A_+ &= \frac{\phi_0}{2} - \frac{\psi_0}{2\sqrt{2}} \,, & A_- &= \frac{\phi_0}{2} + \frac{\psi_0}{2\sqrt{2}} \,, \\ B_+ &= \frac{\dot\phi_0}{2\omega_+} - \frac{\dot\psi_0}{2\sqrt{2}\,\omega_+} \,, & B_- &= \frac{\dot\phi_0}{2\omega_-} + \frac{\dot\psi_0}{2\sqrt{2}\,\omega_-} \,. \end{aligned}$$