1A. Calculate

$$I \;=\; \int_{-\pi}^{+\pi} d\phi\,\delta\left[\phi\,\left(\phi^2-\frac{\pi^2}{4}\right)\right]\,\cos(\phi)\,.$$

Solution:

$$\begin{aligned}
I \;&=\; \int_{-\pi}^{+\pi} d\phi\,\cos(\phi)\,\delta\left[\phi\,(\phi-\pi/2)(\phi+\pi/2)\right] \\
&=\; \int_{-\pi}^{+\pi} d\phi\,\cos(\phi) \\
&\quad\;\; \left[\frac{\delta(\phi)}{\pi^2/4}+\frac{\delta(\phi-\pi/2)}{\pi^2/2}+\frac{\delta(\phi+\pi/2)}{\pi^2/2}\right] \\
&=\; \frac{4}{\pi^2}\,\cos(0)+\frac{2}{\pi^2}\,\cos\left(\frac{\pi}{2}\right)+\frac{2}{\pi^2}\,\cos\left(-\frac{\pi}{2}\right) \\
&=\; \frac{4}{\pi^2}\,.
\end{aligned}$$

1B. Calculate

$$I \;=\; \int_{-\pi}^{+\pi} d\phi\,\delta\left[\phi\,\left(\phi^2-\frac{\pi^2}{16}\right)\right]\,\cos(\phi)\,.$$

Solution:

$$
\begin{aligned}
I &= \int_{-\pi}^{+\pi} d\phi \, \cos(\phi) \, \delta\left[\phi\,(\phi - \pi/4)(\phi + \pi/4)\right] \\
&= \int_{-\pi}^{+\pi} d\phi \, \cos(\phi) \\
&\quad \left[\frac{\delta(\phi)}{\pi^2/16} + \frac{\delta(\phi - \pi/4)}{\pi^2/8} + \frac{\delta(\phi + \pi/4)}{\pi^2/8}\right] \\
&= \frac{16}{\pi^2} \cos(0) + \frac{8}{\pi^2} \cos\left(\frac{\pi}{4}\right) + \frac{8}{\pi^2} \cos\left(-\frac{\pi}{4}\right) \\
&= \frac{16 + 8\sqrt{2}}{\pi^2} \, .
\end{aligned}
$$

2. The total charge on the infinitely thin conducting surface of a sphere of radius $r_0$ is $Q$. Write down the charge density $\rho(r, \theta, \phi)$.

Solution:

$$
\rho(r, \theta, \phi) = \rho(r) = \frac{Q}{4\pi\, r_0} \, \delta(r - r_0)
$$

so that

$$
\int_{-1}^{+1} d\cos(\theta) \int_0^{2\pi} d\phi \int_0^{\infty} r^2 \, dr \, \rho(r) = Q \, .
$$