Mathematical Physics — PHZ 3113
Solutions Midterm 1 (February 18, 2013)

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1. Calculate the gradient of the 3D potential (in arbitrary units) $\frac{1}{2}\,r^2$.

   Solution (with Einstein convention):

$$\nabla \frac{1}{2} r^2 = r\,\nabla\, r = r\,\hat{x}_i\,\partial_i\, r = r\,\hat{x}_i\,\frac{x_i}{r}$$
$$= r\,\frac{\vec{r}}{r} = r\,\hat{r} = \vec{r}\,.$$

2. A force in 3D is (in arbitrary units) given by $\vec{F} = -r\,\vec{r}$. Use (with Einstein convention)

$$\nabla \times r\,\vec{r} = \epsilon_{ijk}\,\hat{x}_i\,\partial_j\, r\, x_k$$

   to calculate the curl of this force.

   Solution:

$$\epsilon_{ijk}\,\hat{x}_i\,\partial_j\, r\, x_k =$$
$$\epsilon_{ijk}\,\hat{x}_i\, x_k\,\partial_j\, r + \epsilon_{ijk}\,\hat{x}_i\, r\,\partial_j\, x_k =$$

$$\epsilon_{ijk}\,\hat{x}_i\,x_k\,\frac{x_j}{r} + \epsilon_{ijk}\,\hat{x}_i\,r\,\delta_{j\,k} =$$

$$\frac{1}{r}\,\epsilon_{ijk}\,\hat{x}_i\,x_j\,x_k + \epsilon_{ijj}\,\hat{x}_i\,r = 0+0 = 0\,.$$

The first term is zero, because it is the contraction of an anti-symmetric with a symmetric tensor, the second because of $\epsilon_{ijj} = 0$.

3. Calculate the curl of the force

$$\vec{F} = x_2\,\hat{x}_1 + x_3\,\hat{x}_2 + x_1\,\hat{x}_3 = \begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix}\,.$$

Solution:

$$\begin{aligned} \nabla \times \vec{F} &= \hat{x}_1\,(\partial_2\,F_3 - \partial_3\,F_2) \\ &\quad + \hat{x}_2\,(\partial_3\,F_1 - \partial_1\,F_3) \\ &\quad + \hat{x}_3\,(\partial_1\,F_2 - \partial_2\,F_1) \\ &= \hat{x}_1\,(0-1) + \hat{x}_2\,(0-1) + \hat{x}_3\,(0-1) \\ &= -\hat{x}_1 - \hat{x}_2 - \hat{x}_3 = -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\,. \end{aligned}$$

4. A point mass on an inclined plane experiences under gravity a force $\vec{F} = -F\,\hat{z}$ in downward direction (see the figure). Find the magnitudes of the forces perpendicular and parallel to the plane as function of the angle α.

   Solution:

   $$F_\perp = F\,\cos(\alpha)\,, \qquad F_\| = F\,\sin(\alpha)\,.$$

5. A point mass is suspended as shown in the figure. As in the previous problem it experiences a force $\vec{F} = -F\,\hat{z}$ in downward direction. Find the z components of the tensions $\vec{T}^{\,1}$ and $\vec{T}^{\,2}$ as functions of the angles α_1 and α_2. In which limit becomes T^1_x infinite?

   Solution: There are four equations for the four unknowns T^1_x, T^1_z, T^2_x and T^2_z as functions of three free parameters F, α_1, α_2:

$$
\begin{aligned}
T_z^1 + T_z^2 &= F\,, \\
T_x^1 + T_x^2 &= 0\,, \\
\tan\alpha_1 &= \frac{T_z^1}{T_x^1}\,, \\
\tan\alpha_2 &= \frac{T_z^2}{-T_x^2} = \frac{T_z^2}{T_x^1}\,.
\end{aligned}
\tag{1}
$$

Now,

$$
\frac{\tan\alpha_1}{\tan\alpha_2} = \frac{T_z^1}{T_z^2}\,, \qquad T_z^2 = \frac{\tan\alpha_2}{\tan\alpha_1}\, T_z^1\,,
$$

$$
T_z^1\left(1 + \frac{\tan\alpha_2}{\tan\alpha_1}\right) = F\,, \quad T_z^1 = \frac{F\,\tan\alpha_1}{\tan\alpha_1 + \tan\alpha_2}\,,
$$

$$
T_z^2\left(1 + \frac{\tan\alpha_1}{\tan\alpha_2}\right) = F\,, \quad T_z^2 = \frac{F\,\tan\alpha_2}{\tan\alpha_1 + \tan\alpha_2}\,.
$$

It follows from (1) that

$$
T_x^1 = T_z^1\,\tan\alpha_1 = \frac{F}{\tan\alpha_1 + \tan\alpha_2}
$$

becomes infinite for $\alpha_1 \to 0$ and $\alpha_2 \to 0$.

Second solution:

$$T^1 \, \sin\alpha_1 + T^2 \, \sin\alpha_2 \, = \, F \, ,$$
$$T^1 \, \cos\alpha_1 - T^2 \, \cos\alpha_2 \, = \, 0 \, .$$

Solving for T^1:

$$T^2 \, = \, T^1 \, \frac{\cos\alpha_1}{\cos\alpha_2} \, ,$$
$$F \, = \, T^1 \, \left(\sin\alpha_1 + \sin\alpha_2 \frac{\cos\alpha_1}{\cos\alpha_2} \right) \, ,$$
$$T^1 \, = \, F \, \frac{\cos\alpha_2}{\sin\alpha_1 \, \cos\alpha_2 + \cos\alpha_1 \, \sin\alpha_2} \, ,$$

$$T^1 \, = \, F \, \frac{\cos\alpha_2}{\sin(\alpha_1 + \alpha_2)} \, . \qquad (2)$$

Solving for T^2:

$$T^1 \, = \, T^2 \, \frac{\cos\alpha_2}{\cos\alpha_1} \, ,$$
$$F \, = \, T^2 \, \left(\sin\alpha_2 + \sin\alpha_1 \frac{\cos\alpha_2}{\cos\alpha_1} \right) \, ,$$
$$T^2 \, = \, F \, \frac{\cos\alpha_1}{\sin\alpha_2 \, \cos\alpha_1 + \cos\alpha_2 \, \sin\alpha_1} \, ,$$

$$T^2 \, = \, F \, \frac{\cos\alpha_1}{\sin(\alpha_1 + \alpha_2)} \, . \qquad (3)$$

The components follow from

$$
\begin{aligned}
T_x^1 &= T^1 \cos\alpha_1 \;=\; -T_x^2\,, \\
T_z^1 &= T^1 \sin\alpha_1\,, \\
T_z^2 &= T^2 \sin\alpha_2\,.
\end{aligned}
$$