Mathematical Physics — PHZ 3113

Midterm 2 (March 18, 2013)

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1. Use the definition $\nabla \times \vec{A} = \epsilon_{ijk}\hat{x}_i\partial_j A_k$ (with Einstein convention) and properties of the Levi-Civita tensor ϵ_{ijk} to transform
$$\nabla \times \nabla \times \vec{A}$$
into applications of the ∇ operator, which do no longer involve the curl.

2. Calculate the volume of an ellipsoid
$$V = \int_{\left(\frac{x_1}{a}\right)^2+\left(\frac{x_2}{b}\right)^2+\left(\frac{x_3}{c}\right)^2\leq 1} d^3x \,.$$
Hint: With suitable substitutions the integral can be mapped on the integral for the volume of a sphere.

3. Calculate
$$\vec{v} = \vec{\omega} \times \vec{r}$$
and express the result in spherical coordinates. Here $\vec{\omega} = \dot{\phi}\,\hat{z}$ is the angular velocity and $\vec{r} = r\,\hat{r}$ the position vector.

4. Calculate the time derivative
$$\dot{\hat{r}} = \frac{d\,\hat{r}}{d\,t}$$
of the unit vector
$$\hat{r} = \frac{\vec{r}}{|\vec{r}|}$$
and express the result in spherical coordinates.

5. The orbit of a planet is assumed to be an ellipse, which in cylindrical coordinates is given by the equation

$$\rho \;=\; \frac{p}{1+\epsilon\,\cos\phi}$$

where $0 \le \epsilon < 1$ is the eccentricity and p the latus rectum. From the analytical solution of Kepler's problem it follows that the magnitude of the angular momentum is (up to a correction for the finite sun mass) given by

$$L = \mathrm{const}\, m\, \sqrt{p}\,,$$

where m is the mass of the planet. Use this information and angular momentum conservation to derive Kepler's 3rd law.