Mathematical Physics — PHZ 3113

Solutions Midterm 2 (March 18, 2013)

---

1. Use the definition $\nabla \times \vec{A} = \epsilon_{ijk} \hat{x}_i \partial_j A_k$ (with Einstein convention) and properties of the Levi-Civita tensor ϵ_{ijk} to transform

$$\nabla \times \nabla \times \vec{A}$$

into applications of the ∇ operator, which do no longer involve the curl.

Solution:

$$\begin{aligned}\nabla \times \nabla \times \vec{A} &= \epsilon_{ijk} \hat{x}_i \partial_j \epsilon_{klm} \partial_l A_m \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\ \hat{x}_i \partial_j \partial_l A_m \\ &= \hat{x}_i \partial_j \partial_i A_j - \hat{x}_i \partial_j \partial_j A_i \\ &= (\hat{x}_i \partial_i)(\partial_j A_j) - (\partial_j \partial_j)(\hat{x}_i A_i) \\ &= \nabla \left(\nabla \cdot \vec{A}\right) - \nabla^2 \vec{A} .\end{aligned}$$

2. Calculate the volume of an ellipsoid

$$V = \int_{\left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 + \left(\frac{x_3}{c}\right)^2 \le 1} d^3x .$$

Hint: With suitable substitutions the integral can be mapped on the integral for the volume of a sphere.

Solution: Let $x_1 = a\,x'_1$, $x_2 = b\,x'_2$ and $x_3 = c\,x'_3$. With these substitution we find

$$\begin{aligned} V &= a\,b\,c \int_{x_1'^{\,2}+x_2'^{\,2}+x_3'^{\,2}\leq 1} d^3x' \\ &= a\,b\,c \int_0^1 r'^{\,2}\,dr' \int_{-1}^{+1} d\cos\theta' \int_0^{2\pi} d\phi' \\ &= \frac{4\pi}{3}\,a\,b\,c\,. \end{aligned}$$

3. Calculate

$$\vec{v} = \vec{\omega} \times \vec{r}$$

and express the result in spherical coordinates. Here $\vec{\omega} = \dot{\phi}\,\hat{z}$ is the angular velocity and $\vec{r} = r\,\hat{r}$ the position vector.

Solution:

$$\vec{v} = \dot{\phi}\,r\,\sin(\theta)\,\hat{\phi} = \dot{\phi}\,\rho\,\hat{\phi}\,.$$

4. Calculate the time derivative

$$\dot{\hat{r}} = \frac{d\,\hat{r}}{d\,t}$$

of the unit vector

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|}$$

and express the result in spherical coordinates.

Solution: From

$$\begin{aligned}
\hat{r} &= \sin(\theta)\ \cos(\phi)\,\hat{x} + \sin(\theta)\ \sin(\phi)\,\hat{y} + \cos(\theta)\,\hat{z}\,,\\
\hat{\theta} &= \cos(\theta)\ \cos(\phi)\,\hat{x} + \cos(\theta)\ \sin(\phi)\,\hat{y} - \sin(\theta)\,\hat{z}\,,\\
\hat{\phi} &= -\sin(\phi)\,\hat{x} + \cos(\phi)\,\hat{y}
\end{aligned}$$

we get

$$\begin{aligned}
\frac{\partial\,\hat{r}}{\partial\,r} &= 0\,,\\
\frac{\partial\,\hat{r}}{\partial\,\theta} &= \cos(\theta)\cos(\phi)\,\hat{x} + \cos(\theta)\sin(\phi)\,\hat{y}\\
&- \sin(\theta)\,\hat{z} \;=\; \hat{\theta}\,,\\
\frac{\partial\,\hat{r}}{\partial\,\phi} &= -\sin(\theta)\ \sin(\phi)\,\hat{x} + \sin(\theta)\ \cos(\phi)\,\hat{y}\\
&= \sin(\theta)\,\hat{\phi}\,.
\end{aligned}$$

Therefore,

$$\dot{\hat{r}} \;=\; \frac{\partial\,\hat{r}}{\partial\,\theta}\,\dot{\theta} + \frac{\partial\,\hat{r}}{\partial\,\phi}\,\dot{\phi} \;=\; \dot{\theta}\,\hat{\theta} + \sin(\theta)\,\dot{\phi}\,\hat{\phi}\,.$$

5. The orbit of a planet is assumed to be an ellipse, which in cylindrical coordinates is given by the equation

$$\rho \;=\; \frac{p}{1+\epsilon\ \cos\phi}$$

where $0 \le \epsilon < 1$ is the eccentricity and p the latus rectum. From the analytical solution of Kepler's problem it follows that the magnitude of the angular momentum is (up to a correction for the finite sun mass) given by

$$L = \text{const}\, m\,\sqrt{p}\,,$$

where m is the mass of the planet. Use this information and angular momentum conservation to derive Kepler's 3rd law.

Solution: For $\phi = 0$ we have (a major half-axis and ϵ eccentricity)

$$a - \epsilon\, a = \frac{p}{1+\epsilon} \;\Rightarrow\; p \;=\; a\,(1-\epsilon^2)\,.$$

Angular momentum conservation implies

$$\frac{L}{m} \;=\; \rho^2\,\dot{\phi} \;=\; \text{const}\,\sqrt{a\,(1-\epsilon^2)}$$
$$\int_0^{2\pi} \rho^2\,d\phi \;=\; \text{const}\,\sqrt{a\,(1-\epsilon^2)}\,\int_0^T dt$$

where T is the orbital period. As

$$df = \frac{1}{2}\,\rho^2\,d\phi$$

is the infinitesimal sectorial area of an ellipse, the integral on the left side is two times the area of an ellipse, $2\pi\,a\,b$, where $b = a\sqrt{1-\epsilon^2}$ is the minor axis of the ellipse. Collecting the terms we have

$$2\pi\,a^2 \;=\; \text{const}\,\sqrt{a}\,T\,.$$

Squaring both sides, we arrive at Kepler's 3rd law:

$$a^3 \;=\; \text{const}'\,T^2\,,$$

where const$'$ is a new constant.