"Sensible mathematics involves neglecting a quantity when it is small, not because it is infinitely great and we do not want it."

- P.A.M. Dirac

**Dirac Cosmology**

\[
\frac{hC}{\text{cm} \text{m} \text{s}^{-1}} \sim \frac{\text{Mpc}}{h \cdot H} \sim 10^{41}
\]

Speculation: \( h \approx 1/6 \)

\( \Rightarrow H = 160 \text{ km/s/Mpc} \)

Only a factor of two off!
Cosmology for Beginners

The Metric

Minkowski Space (Special Relativity)

\[ ds^2 = dt^2 - dx^2 = \text{proper time} \]

\[ = \sum_{\mu, \nu = 0}^3 \eta_{\mu \nu} dx^\mu dx^\nu \]

\[ \eta_{\mu \nu} = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix} \]

Protons travel on paths of zero length, called null geodesics:

\[ ds^2 = 0 \]

Massive particles travel on timelike geodesics:

\[ ds^2 > 0 \]
Cosmology For Beginners

Einstein Field Equation

Metric in General Relativity depends on space and time. Determined by Einstein Field Equation

\[ G_{\mu \nu} = 8 \pi G T_{\mu \nu} \]

- Function of metric and derivatives
- Newton's constant
- Stress-energy of matter

Analogies: Electromagnetism

Maxwell's Equation

\[ d\mathbf{F}^\mu = \frac{4\pi}{c} \mathbf{J}^\mu \]

- Field Tensor
- Charge and current density
Cosmology For Beginners

Einstein Field Equation

- EFE: 6 simultaneous, nonlinear, 2nd-order PDEs of 10 functions. Complicated!

Solution: Assume a symmetry.

1. Vacuum: $T_{\mu \nu} = 0$

   
   \[ \text{EFE} \rightarrow \text{Wave Equation} \]

   - Gravitational waves.

2. Spherical Symmetry

   - Schwarzschild solution
   - Black Holes

3. Homogeneity, Isotropy

   - Robertson-Walker spaces
Robertson–Walker Spaces

General homogeneous, isotropic metric:

\[ ds^2 = dt^2 - a^2(t) d\chi^2 \]

\[ = dt^2 - a^2(t) \left[ \frac{dr^2}{1-k r^2} - r^2 d\Omega^2 \right] \]

\[ k = 0, \pm 1 \]

= curvature

Einstein Field Equations

\((\ddot{a}/a)^2 + k/a^2 = \frac{8\pi}{3m_{\text{pl}}^2} \rho \quad (m_{\text{pl}}^2 = 1/4G)\)

\(\ddot{a}/a = -\frac{4\pi}{3m_{\text{pl}}} \left( \rho + 3\pi \right) \checkmark \) Pressure gravitates!

Stress–Energy Conservation

\(\dot{\rho} + 3 \left( \frac{\dot{a}}{a} \right) (\rho + \pi) = 0\)
Photon Redshift

One effect of expansion is that photons' redshift is proportional to the scale factor:

\[ \lambda \propto a(t) \]

or, in terms of the redshift factor \( z \),

\[ (1+z) \equiv \frac{\lambda_0}{\lambda_{em}} = \frac{a(t)}{a(t_{em})} \]

NOTE: This is not a "Doppler shift" because of recession velocity!
Robertson—Walker Spaces

**Curvature**

\[ k = 0 \rightarrow \text{flat} \]
\[ k = 1 \rightarrow \text{positive curvature} \]
\[ k = -1 \rightarrow \text{negative curvature} \]

**Critical Density**

For a given expansion rate,

\[ H = \left( \frac{\ddot{a}}{a} \right) \]

\( \rho_c \) is the density such that \( k = 0 \):

\[ H^2 = \left( \frac{\ddot{a}}{a} \right)^2 = \frac{8 \pi}{3 m_{pl}^2} \rho_c \]

\[ \Rightarrow \quad \rho_c = \frac{3 m_{pl}^2}{8 \pi} H^2 \]
**Density Parameter**

\[ \Omega = \frac{\rho}{\rho_c} = \frac{8\pi}{3m_p^2} \left( \frac{\rho}{H^2} \right) \]

**Friedmann Equation**

\[ \Omega = 1 + \frac{k}{(cH)^2} \]

- \[ \Omega = 1 \implies k = 0 \quad \text{(flat)} \]
- \[ \Omega > 1 \implies k = 1 \quad \text{(closed)} \]
- \[ \Omega < 1 \implies k = -1 \quad \text{(open)} \]
The Three Kinds of Energy

Continuity Equation
\[ \dot{p} + 3 \left( \frac{\dot{a}}{a} \right) (p + p_r) = 0 \]

Matter
\[ p_m \propto a^{-3} \Rightarrow p_m = \varnothing \]
\[ a(t) \propto t^{2/3} \quad \leftarrow \text{flat universe} \]

Radiation
\[ p_r \propto a^{-4} \Rightarrow p_r = \frac{1}{3} p \varnothing \]
\[ a(t) \propto t^{1/2} \]

Vacuum (Cos. Const.)
\[ p_\varnothing = \text{const.} \Rightarrow p_\varnothing = -p_\varnothing \]
\[ a(t) \propto e^{Ht} \]
\[ H = \sqrt{\frac{8\pi G}{3}} p_\varnothing = \text{const.} \]
THE THREE KINDS OF ENERGY

General Case

Equation of state

\[ p = w \rho \quad w = \text{const.} \]

\[ \Rightarrow \rho \propto a^{-3(1+w)} \]

Flat Universe

\[ \left( \frac{\dot{a}}{a} \right)^2 \propto p \propto a^{-3(1+w)} \]

\[ \Rightarrow a(t) \propto t^{2/3(1+w)} \]

How does a non-flat universe evolve?
HORIZON PROBLEM

The Cosmological Horizon

The universe has a finite age
⇒ photons can only travel a finite distance in time
since Big Bang.
⇒ Universe has a horizon.

Photons travel on null geodesics,
\[ ds^2 = dt^2 - a^2(t) \left| d\vec{x} \right|^2 = 0 \]

⇒ Distance \( \left| d\vec{x} \right| \) covered by a photon

\[ \left| d\vec{x} \right| = \frac{dt}{a(t)} \]

Horizon Size

\[ d_H(t) = \int_0^t \frac{dt'}{a(t')} \]
FLATNESS PROBLEM

Friedmann Equation

\[ w = \text{const.} \implies \]

\[
\frac{d \Omega}{d \log(a)} = (1 + 3w) \Omega (\Omega - 1)
\]

Exact for \( k \neq \emptyset \)!

\[ p = \emptyset \text{ (Matter)} \quad p = p/3 \text{ (Radiation)} \]

\[ (1 + 3w) > \emptyset \]

Universe is an unstable fixed point:

\[
\frac{d |\Omega - 1|}{d \ln(a)} > \emptyset \quad (1 + 3w) > \emptyset
\]

Flatness Problem
FLATNESS PROBLEM

The universe evolves away from flatness:

\[ \Omega \]

1

\[ t \]

TODAY: \( \Omega_0 = 1 \pm 0.05 \)

(Actual WMAP 5: \( \Omega_0 = 0.99^{+0.100}_{-0.085} \))

Recombination: \( \Omega = 1 \pm 0.00004 \)

Nucleosynthesis: \( \Omega = 1 \pm 10^{-12} \)
Horizon Problem

Conformal Coordinates

Take the FRW metric

$$ds^2 = dt^2 - a^2(t) \left(1 - \frac{\kappa}{a^2} \right)^2$$

and re-write in terms of the conformal time $d\tau$:

$$ds^2 = a^2(\tau) \left[ d\tau^2 - \left(1 - \frac{\kappa}{a^2} \right) d\bar{x}^2 \right]$$

where a conformal "clock" slows down with the expansion of the universe,

$$d\tau = \frac{dt}{a(t)}$$

Then the horizon size is

$$d_H = \int_0^t \frac{dt'}{a(t')} = \int_0^\infty d\tau^1 = \infty$$
Horizon Problem

In conformal coordinates, photon geodesics are $ds^2 = 0 \Rightarrow dx = dz$, or (conformally) Minkowski. So we can make a space-time diagram of an expanding universe with light travelling at $45^\circ$ angles. (Expansion is factored out).

- Light cone is cut off at the Big Bang, $t = 0$
- $dt$ is just the width of the light cone.
Horizon Problem

KEY POINT: Two events on the conformal diagram are causally connected if their past light cones intersect.

Now look at two points 180° apart on the CMB:

Causally disconnected! How did they get in thermal equilibrium?

Horizon Problem
Exercise

Show that for $w = \text{const.}$,

$$\left(\frac{d}{d\Omega}\right)^2 |\Omega - 1| = \text{const.}$$

where $d$ is some comoving distance, $d \propto a(t)$.

Therefore, for a universe evolving away from flatness,

$$\frac{d |\Omega - 1|}{d \log(a)} > 0$$

The horizon size gets bigger in comoving coordinates

$$\frac{d}{d \log(a)} \left(\frac{d}{d\Omega}\right) < 0$$

i.e., more and more space "falls into" horizon, or becomes causally connected, at late times.
SOLUTION TO THE HORIZON/PLANENESS PROBS

\[ \frac{d\Omega}{d\log(a)} = (1+3w) \Omega (\Omega^{-1}) \]

\[ \Rightarrow \frac{d(\Omega^{-1})}{d\log(a)} \propto (1+3w) \begin{cases} > 0 & w > -\frac{1}{3} \\ < 0 & w < -\frac{1}{3} \end{cases} \]

For \( w < -\frac{1}{3} \), universe evolves baryon and flatness \( \Rightarrow \text{solves F. P.}! \) (NEGATIVE PRESSURE!)

Similarly for the horizon problem:

\[ \frac{d}{d\log(a)} \left( \frac{d}{dt} \right) \propto -(1+3w) \begin{cases} < 0 & w > -\frac{1}{3} \\ > 0 & w < -\frac{1}{3} \end{cases} \]

\[ \Rightarrow \text{horizon shrinks in comoving units, and comoving scales are "redshifted out of horizon".} \]

(animations)

RW EQ:

\[ \left( \frac{\ddot{a}}{a} \right) \propto -(1+3w) \Rightarrow w < -\frac{1}{3} \text{ means accelerating expansion} . \]
NEGATIVE PRESSURE

Positive Pressure: atoms bounce off walls of container, pushing out

\[ p = - \frac{du}{dV} > 0 \]

Vacuum: energy density is constant, so free energy is proportional to volume.

\[ u \propto V \Rightarrow p = - \frac{du}{dV} = -1 \]

Example: rubber band:

\[ \frac{du}{dx} > 0 \Rightarrow p < 0 \]

Any system in tension (common in solid state) has negative pressure.
Horizon Problem in Conformal Coords.

Take the extreme case of a cosmological constant:

\[ P_m = \text{const.} \Rightarrow \rho = -p \]

Friedmann Equation

\[ H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} P_m = \text{const.} \]

\[ \Rightarrow \quad a \propto e^{H t} : \text{Exponential Expansion} \]

Conformal Time:

\[ dx = \frac{dt}{a(t)} = e^{-Ht} dt \]

\[ \Rightarrow \]

\[ \tau = -\frac{1}{H} e^{-Ht} = -\frac{1}{dt} \]

Negative Conformal Time: \( \tau \to -\infty \) at late time.
Horizon Problem in Conformal Coords.

Big Bang is pushed to negative conformal time: \( \tau = 0 \) marks the end of inflation.

\[
\begin{array}{c}
\tau = 0 \\
\end{array}
\]

Note: Approximation around this point breaks down near \( \tau = 0 \): equation of state not constant.
**INFLATION FROM SCALAR FIELDS**

**Action for minimally coupled scalar field**

\[
S = \int \left( \frac{1}{\sqrt{-g}} \left[ R + \mathcal{L}_\phi \right] \right) \, d^4x
\]

*From wedge product*

\[
dx^0 \wedge \cdots \wedge dx^3 = \sqrt{-g} \, dx^0 \cdots dx^3
\]

\[
\sqrt{-g} = \frac{1}{\sqrt{-\text{Det}(g_{\mu\nu})}}
\]

*Minimally coupled = no R \phi terms.*

\[
\mathcal{L}_\phi = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi)
\]

**Flat FRW metric**

\[
g_{\mu\nu} = a^2(t) \, \eta_{\mu\nu} \quad [\text{Conformal coords}]
\]

\[
= \begin{pmatrix}
1 & -a^3 \\
-a^3 & -a^2 \\
-a^3 & -a^2
\end{pmatrix} \quad [\text{Proper time}]
\]
Equation of motion (FRW)

\[ \ddot{\Phi} + 3H \dot{\Phi} - \nabla^2 \Phi + \frac{\delta V}{\delta \Phi} = 0 \]

Stress-energy

\[ T^{\mu \nu} = \text{d}^\mu \Phi \text{d}^\nu \Phi - g^{\mu \nu} L \]

Homogeneous mode \( \vec{k} = 0 \Rightarrow \vec{V} \Phi_{\vec{k}} = 0 \)

\[ \Rightarrow \ddot{\Phi} + 3H \dot{\Phi} + V'(\Phi) = 0 \]

\[ T^{\mu \nu} = \begin{pmatrix} p & -p & -p \\ -p & p & -p \\ -p & -p & p \end{pmatrix} \]

\[ p = \frac{1}{2} \dot{\Phi}^2 + V(\Phi) \]

\[ p = \frac{1}{2} \dot{\Phi}^2 - V(\Phi) \]
**Inflation from Scalar Fields**

**Slow Roll**

Inflation ($\dot{a} > 0$) requires $P_{\phi} >> \dot{\phi}^2 << V(\phi)$

Called slow roll.

---

**Cosmological Equations of Motion**

\[
H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3m^2_{pl}} \left[ \frac{1}{2} \dot{\phi}^2 + V(\phi) \right]
\]

\[
\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0
\]

Therefore,

\[
H^2 = \frac{8\pi}{3m^2_{pl}} V(\phi) = H(\phi)
\]

and

\[
a(t) \propto \exp \left[ \int H dt \right] = e^{-N}
\]
INFLATION FROM SCALAR FIELDS

Slow Roll Parameters

We can re-write the acceleration equation in a convenient form:

\[
\left( \frac{\ddot{a}}{a} \right) = -\frac{4\pi}{3 m_p^2} (\rho + 3P) = H^2 (1 - \epsilon)
\]

\[
\epsilon = \frac{3}{2} \left( \frac{P}{\rho} + 1 \right)
\]

So that the condition for inflation is:

\[
\ddot{a} > 0 \Rightarrow \epsilon < 1
\]

It is straightforward to show:

\[
\epsilon = -\frac{d}{d \ln a} \frac{d \ln H}{d \ln a} = \frac{1}{H} \frac{dH}{dN}
\]

where \( N \) is the number of e-folds defined above.
INFLATION FROM SCALAR FIELDS

**Slow Roll Parameters**

In the limit of slow roll,

\[ H^2(\phi) \approx \frac{8 \pi}{3 m_{pl}^2} V(\phi) \]

Then the number of e-folds is:

\[ N = - \int H \, dt = - \int \frac{H}{\dot{\phi}} \, d\phi \]

\[ \approx \frac{8 \pi}{m_{pl}^2} \int_0^{\Phi_e} \frac{V(\phi)}{V'(\phi)} \, d\phi \quad \text{Count backward from } \Phi_e ! \]

Then

\[ \epsilon = \frac{1}{H} \frac{dH(\phi)}{dN} \approx \frac{m_{pl}^2}{16 \pi} \left( \frac{V'(\phi)}{V(\phi)} \right)^2 \]

For inflation,

\[ \epsilon \ll 1 \Rightarrow V'(\phi) \ll V(\phi) \]

Flat potential!
Basic Thermodynamics of Inflation

Consider a universe filled with junk:

\[ p_m \propto a^{-3}, \quad p_\phi \propto a^{-4}, \quad p_n = \text{const.} \]

If \( p_n > p_m, p_\phi \), then we expand exponentially:

\[ a(t) \propto e^{Ht} \Rightarrow p_m \propto e^{-3Ht} \]
\[ p_\phi \propto e^{-4Ht} \]

Other components are exponentially diluted.

Result is an empty universe!
BASIC THERMODYNAMICS OF INFLATION

REAL UNIVERSE: VACUUM ENERGY IS DYNAMIC.

\( \Rightarrow \) Order parameter with quantum numbers of vacuum field \( \phi \), "Inflaton"

\[ \rho \sim V(\phi) \]

\[ \rho \propto a^{-3} \]

*IN ORDER TO PRODUCE "PRIMORDIAL SOUP" OF FUNDAMENTAL PARTICLES, INFLATION MUST DECAY.*

REHEATING

\[ \rho_n \rightarrow \rho_r \quad \text{(assume radiation)} \]
How much inflation?

Exercise:

Show that the total entropy in the horizon today \( dt \) is of order \( \sim 10^{88} \).

Take an energy density during inflation

\( \rho \sim \Lambda^4 \)

and a horizon size

\( d_h \sim H^{-1} \sim m_p / \Lambda^2 \)

Volume before inflation \( d_V^3 \sim V \).

After inflation

\[ V \sim e^{3N} d_h^3 = e^{3N} \frac{m_p^3}{\Lambda^6} \]

**Key Point:** Entropy density is conserved everywhere but reheating.

\( \Rightarrow \) total entropy is a measure of volume.
How much inflation?

Reheat temp: $T_{RH} \sim \Lambda$ (we will refine this later)

$S_{patch} \sim e^{3N} \frac{m_{pl}^3}{\Lambda^6} T_{RH}^3 \sim e^{3N} \left( \frac{m_{pl}}{\Lambda} \right)^3$

$\geq 10^{88}$

$\Rightarrow \quad N \geq 68 + \ln \left( \frac{\Lambda}{m_{pl}} \right)$

We will see later

$\frac{\Delta p}{p} \sim 10^{-5} \Rightarrow \Lambda \sim 10^{-4} m_{pl}$

$\Rightarrow \quad N \geq 60$

Most inflation models:

$N_{tot} \gg N_{min} \sim 60$

Overkill.
Example: $\lambda \Phi^4$

Take $V(\Phi) = \lambda \Phi^4$

Slow Roll EOM:

$$\dot{\Phi} = -\frac{V'(\Phi)}{3 H} = -\sqrt{\frac{m_{P1}^2}{24\pi}} \frac{V'(\Phi)}{V(\Phi)}$$

End of inflation:

$$\epsilon = \frac{\frac{m_{P1}^3}{16\pi}}{(V(\Phi)^2)} = 1$$

$$= \frac{1}{\pi} \left( \frac{m_{P1}}{\Phi_e} \right)^2 \Rightarrow \Phi_e = \frac{m_{P1}}{\sqrt{\pi}}$$

|| Inflation: $\epsilon < 1$ for $\Phi > \Phi_e$

Number of e-folds before and of inflation:

$$N = \frac{2\sqrt{\pi}}{m_{P1}} \int_{\Phi_e}^{\Phi} \frac{d\Phi'}{\sqrt{V(\Phi')}} = \pi \left( \frac{\Phi}{m_{P1}} \right)^2 - 1$$

Integrate backward from $\Phi_e$

$$\Rightarrow \Phi_0 = m_{P1} \sqrt{\frac{N+1}{\pi}} \Rightarrow \Phi_0 = 4.4 \ m_{P1}$$
PERTURBATIONS

Take an arbitrary free scalar \( \psi \).

Equation of motion

\[
\frac{1}{\sqrt{-g}} \partial_\nu (g^{\mu\nu} \sqrt{-g} \partial_\mu \psi) = 0
\]

\( \Rightarrow \psi'' + 2 \left( \frac{a^i}{a} \right) \psi' - \nabla^2 \psi = 0 \)

(Notation: \( \frac{\partial}{\partial x} \), \( \tau \) = conformal time).

Example: Gravity waves

\( g_{\mu\nu} = g_{\mu\nu}^{RW} + h_{\mu\nu} \)

\( h_{0i} = h_{i0} = 0 \quad h_{ij} = \Psi_+ e^i_{ij} + \Psi_x e^x_{ij} \)

\( \Psi_+ \), \( \Psi_x \) satisfy free scalar EOM.

**Exercise:** Derive the equation of state of gravity waves.
PERTURBATIONS

Fourier expand the field:

\[ \psi(t, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \left[ \psi_k(t) a_k e^{i \mathbf{k} \cdot \mathbf{x}} + \psi_k^*(t) a_k^* e^{-i \mathbf{k} \cdot \mathbf{x}} \right] \]

(Classical case!) NOT SCALE FACTOR!

\[ k = \text{comoving wavenumber} \quad k^{\text{phys}} = k/a \]
\[ x = \text{comoving coordinate} \]

\[ ds^2 = a^2(t_c) \left[ dt^2 - d\mathbf{x}^2 \right] \]

From

\[ \psi_k'' + 2 \left( \frac{a'}{a} \right) \psi_k' + k^2 \psi_k = 0 \]

Change of variables

\[ u_k(t) = a(t) \psi_k(t) \]

\[ u_k'' + \left[ k^2 - \frac{a''(t)}{a(t)} \right] u_k = 0 \]

KLEIN-GORDON EQ IN EXPANDING SPACETIME
Perturbations

Short wavelength limit:

\[ \frac{k^2 \gg a''/a^3}{U_k'' + k^2 U_k = 0} \]

Conformally Minkowski: Important point: this looks like a standard massless KG equation in conformal coordinates, e.g., e \( \propto e \). Only becomes exactly Minkowski in the \( k \to \infty \) limit.

Solutions:

\[ U_k(t) = \left[ A_k e^{-i k t} + B_k e^{i k t} \right] \]

Setting integration constants is key.
PERTURBATIONS

Long wavelength limit:

\[ k^2 \ll a^4/a \]

\[ a_k \eta_k(x) = a_k \eta_k'(x) \]

\[ \Rightarrow \eta_k \propto a \Rightarrow \Phi_k = \text{const.} \]

"Mode freezing": modes with wavelength larger than the horizon have constant, nonzero amplitude.

Next up: quantization.
PERTURBATIONS

Normalization

\[ u_k(z) = \left[ A_k e^{-ikz} + B_k e^{+ikz} \right] \frac{1}{\sqrt{\text{vol}}} \]

Quantization \( \Rightarrow |A_k|^2 - |B_k|^2 = 1 \)

Choice of vacuum

\( \Rightarrow u_k(z) \propto e^{-ikz} \Rightarrow A_k = 1 \)

"Bunch - Davies Vacuum"

(Subsequent of current interest/controversy.)

\( B_k \neq 0 \iff "Transplanck" \) physics

Key Point

Quantization + choice of vacuum specify normalization of the mode function (up to a phase).
Perturbations (Exact Solutions)

The Power Spectrum

Exercise

Show that the vacuum two-point correlation function for $\phi$ is:

$$\langle 0 | \psi(x, \bar{x}) \psi(x, \bar{x}') | 0 \rangle = \int \frac{d^3k}{(2\pi)^3} \left| \frac{u_k}{a} \right|^2 e^{ik \cdot (x - \bar{x}')}$$

Then

$$\langle \phi^2 \rangle_{x = x'} = \int \frac{d^3k}{(2\pi)^3} \left| \frac{u_k}{a} \right|^2$$

$$\equiv \int \frac{dk}{k} P(k)$$

Power spectrum $P(k) = \text{Power per logarithmic interval}$,

$$P(k) = \left( \frac{k^3}{2\pi^2} \right) \left| \frac{u_k}{a} \right|^2$$

$$\rightarrow \left( \frac{\nu}{2\pi} \right)^2 (-k \epsilon) \rightarrow 0$$

= const. !!
Perturbations

Physical Interpretation

Particle Creation at Horizon

(i.e., a Hawking Temperature!)

Analogy: Black Holes:

Cosmological Horizon: virtual pairs "swept out of causal contact":

\[ dh = H^{-1} \]

Hawking Temperature

\[ T_H = \frac{H}{2\pi} \]
PERTURBATIONS IN SLOW ROLL

For $\epsilon \ll 1$ a constant, we can use the Bessel function solutions as approximate solutions to the mode equation,

$$P(k) \propto k^n$$

$$n = 3 - 2\nu = 3 - \frac{3-\epsilon}{1-\epsilon}$$

$$\alpha = -2\epsilon$$

This applies to free scalars such as tensor modes.

Tensor amplitude $P = \left(\frac{H}{2\pi}\right)^2$

Spectral index $P \propto k^{n_t}$

$$n_t = -2\epsilon$$

SLOW ROLL RESULT
For scalar modes

\[ P_R = \left[ \frac{\delta N}{\delta \phi} \right]^2 = \left( \frac{H^2}{2\pi \phi} \right)^2 \]

\[ \propto k^{n-1} \]

\[ n - 1 = 4 \epsilon - 2 \eta \]

\[ \eta = \frac{m_p^2}{4\pi} \frac{V''(\phi)}{V(\phi)} \approx \frac{m_p^2}{8\pi} \left[ \frac{V''(\phi)}{V(\phi)} - \frac{1}{2} \left( \frac{V'(\phi)}{V(\phi)} \right)^2 \right] \]

All of these quantities are evaluated at \( \phi = \phi_N \) where \( N = 60 \) is the number of e-folds before the end of inflation.
Example: $\lambda \Phi^4$

It is now straightforward to calculate the perturbation spectra:

$$P_N = \left. \frac{H^3}{2\pi^2} \right|_{\Phi = \Phi_N}$$

$$= \left. \frac{1}{2\pi} \left( \frac{8\pi}{3m_F} \right)^2 \left( \frac{24\pi^3}{\sqrt{V''(\Phi)}} \right)^{3/2} \right|_{\Phi = \Phi_N}$$

$$= \left. \frac{4 \sqrt{24\pi}}{3m_F^3} \left( \frac{V(\phi)}{\sqrt{V''(\phi)}} \right)^{3/2} \right|_{\Phi = \Phi_N}$$

$$= \left. \frac{\sqrt{24\pi}}{3} \left( \frac{N+1}{\pi} \right) \lambda^{1/2} \approx 10^{-5} \right|_{\Phi = \Phi_N}$$

$$\Rightarrow \lambda \approx 10^{-5}$$

Highly fine-tuned! (Typical of most inflation models).
\textbf{Example:} \lambda \Phi^4

\[ P_{T} = \frac{4 \pi^4}{m_{\phi} \sqrt{\pi}} \bigg|_{\phi = \phi_N} \]

\[ \therefore \frac{P_T}{P_Q} = \frac{16 \frac{4 \pi^4}{m_{\phi}^3 \pi}}{\frac{4 \pi^2 \Phi^2}{4 \pi}} \]

\[ = \frac{6 \frac{4 \pi^4}{m_{\phi}^3 \pi}}{\frac{3 \pi^2 \Phi^2}{4 \pi}} \]

\[ = \frac{m_{\phi}^3}{\pi} \left( \frac{\Phi}{\phi} \right)^2 = 16 \Phi \in (\phi_N) \]

\[ = \frac{16}{\pi} \left( \frac{m_{\phi}}{\phi_N} \right)^2 = \frac{16}{N+1} \]

\[ \approx 0.26 \]
**Example:** \( \lambda \phi^n \)

For the scalar spectral index

\[ n = 1 - 4 \epsilon(\phi_N) + 2 \eta(\phi_N) \]

where

\[ \epsilon(\phi_N) = \frac{1}{N+1} \]

\[ \eta(\phi_N) = \frac{m_{Pl}^2}{8 \pi} \left[ \frac{\sqrt{}}{\sqrt{}} - \frac{1}{2} \left( \frac{\sqrt{}}{\sqrt{}} \right)^2 \right] \]

\[ = \frac{m_{Pl}^2}{8 \pi} \left[ \frac{12}{\phi_N^2} - \frac{8}{\phi_N^3} \right] \]

\[ = \frac{1}{2 \pi} \left( \frac{m_{Pl}}{\phi_N} \right)^2 = \frac{1}{2(\pi N+1)} \]

\[ \Rightarrow n = 1 - \frac{4}{N+1} + \frac{1}{N+1} \]

\[ = 1 - \frac{3}{N+1} \approx 0.95 \]
TYPES OF INFLATION MODEL

KEY POINT: The form of the potential $V(\phi)$ determines the primordial spectra $P_R, P_T$.

Large-Field Models

$V(\phi) = m^3 \phi^3 + \lambda \phi^4$

Observables: $n < 1$ ("red spectrum")

$\Phi_N \sim m_{pl}$

$\Lambda = V'^4$

$\sim M_{cut}$

$\sim 10^{16}$ GeV
TYPES OF INFLATION MODEL

Small Field Models

\[ V(\phi) = (\phi^2 - V^2)^2 \]

\[ V(0) \]

\[ \Lambda = M_{\text{Planck}} \]

Observables: \[ n < 1 \ (\text{red spectrum}) \]
\[ r \lesssim 0.01 \]

Other examples

\[ V(\phi) = \Lambda^4 \left[ 1 + \cos \left( \frac{\phi}{\Lambda} \right) \right] \in \text{PM4B} \]
("Natural" inflation)

\[ V(\phi) \propto \phi^4 \ln(\phi^2) \]
\( \text{Coleman Weinberg} \)
Types of Inflation Model

Hybrid Inflation

\[ V(\phi) = (\phi^2 + V^2)^2 \]

- Requires an auxiliary field to end inflation.

Observables:
- \( n > 1 \) ("blue" spectrum)
- \( r \ll 0.01 \)

- Can also inflate for \( D \gg \mu_1 \) (large-field limit).