Bayesian Methods: Theory and Practice
Lecture 3.1 – Example Calculations

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Outline

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Introduction

D0 1995 Top Discovery Data

N = 17 events

b_0 = 3.8 ± 0.6 events

Example Calculations

1. Compute the ***p-value***, without and with the background uncertainty.
2. Compute the ***posterior density*** $p(s|N)$ and its mean and standard deviation.
3. Compute ratio of ***evidences*** $p(N|H_1)/p(N|H_0)$.

Introduction – Models

Likelihood Functions

$$p(N|b+s, H_1) = \text{Poisson}(N|b+s) = \exp[-(b+s)]\,(b+s)^N / N!$$

$$p(N|b, H_0) = \text{Poisson}(N|b) = \exp[-b]\,b^N / N!$$

Prior Density

$$\pi(b, s) = \pi(b|s)\,\pi(s)$$

$$\pi(b|s) = \pi(b) = \text{Gamma}(kb|B+1) = k\exp(-kb)\,(kb)^B / \Gamma(B+1)$$

Effective Scale Factor k and Count B

$$b = B / k \qquad B = (b / \delta b)^2 = (3.8 / 0.6)^2 = 41.11$$

$$\delta b = \sqrt{B} / k \qquad k = b / \delta b^2 = 3.8 / 0.6^2 = 10.56$$

Introduction – Integrated Likelihoods

The prior predictive distributions, or **integrated likelihoods**, are

$$p(n \mid s, H_1) = \int_0^\infty \text{Poisson}(n \mid b+s)\text{Gamma}(kb \mid B+1)\,db$$

$$= \left(\frac{k}{1+k}\right)^{B+1} \sum_{r=0}^{n} \frac{1}{(1+k)^r} \frac{\Gamma(B+1+r)}{\Gamma(B+1)r!} \text{Poisson}(n-r \mid s)$$

and

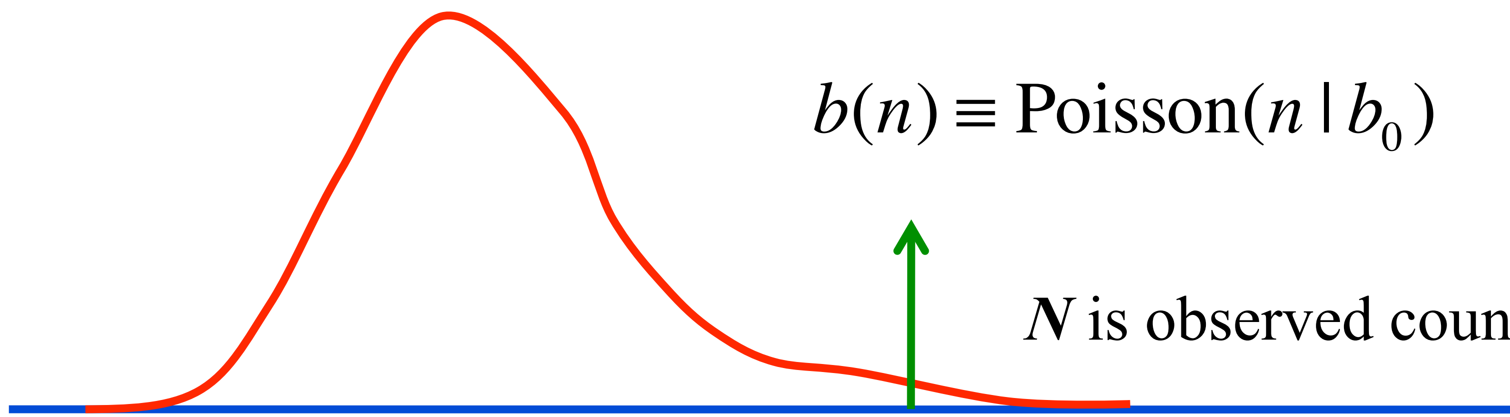
$$p(n \mid H_0) = p(n \mid s=0, H_1) = \left(\frac{k}{1+k}\right)^{B+1} \frac{1}{(1+k)^n} \frac{\Gamma(B+1+n)}{\Gamma(B+1)n!}$$

Example Calculations

Example 1: p-value (a)

Background-only, $b_0 = 3.8$ events (ignoring uncertainty)

$$b(n) \equiv \text{Poisson}(n \mid b_0)$$



N is observed count

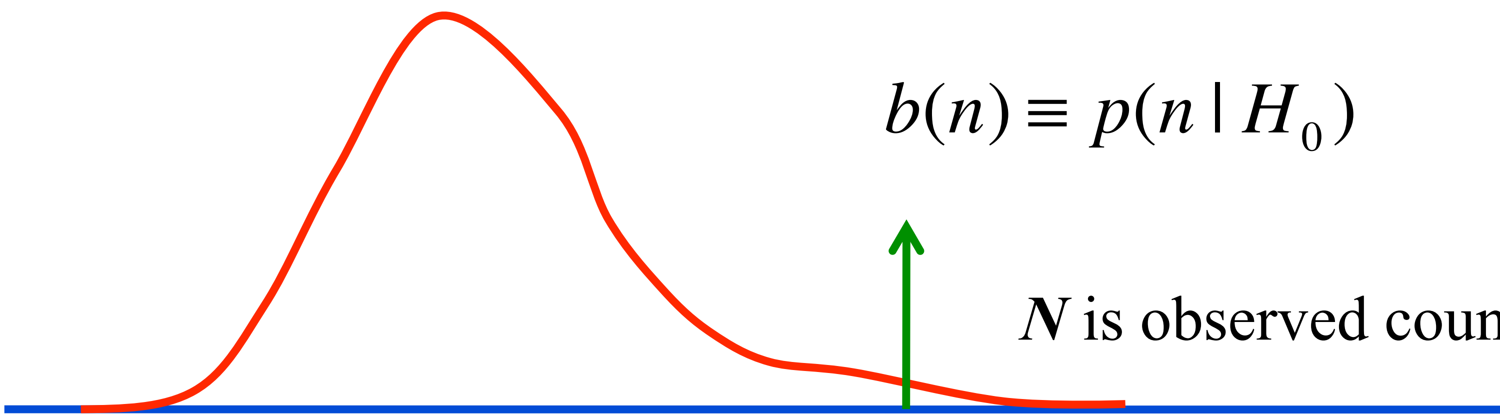
$$N = 17$$

$$p - value = \sum_{n=N}^{\infty} b(n) = \sum_{n=17}^{\infty} \text{Poisson}(n \mid 3.8) = 5.7 \times 10^{-7}$$

This is equivalent to $4.9\ \sigma$

Example 1: p-value (b)

Background-only, $b_0 = 3.8 \pm 0.6$ events



$$b(n) \equiv p(n \mid H_0)$$

N is observed count

$$N = 17$$

$$p - value = \sum_{n=N}^{\infty} b(n) = \sum_{n=17}^{\infty} p(n \mid H_0) = 5.4 \times 10^{-6}$$

This is equivalent to $4.4\ \sigma$

Example 2: Posterior Density

Given the integrated likelihood

$$p(n \mid s, H_1) = \left(\frac{k}{1+k}\right)^{B+1} \sum_{r=0}^{n} c_r(k, B)\,\text{Poisson}(n - r \mid s)$$

where

$$c_r(k, B) \equiv \frac{1}{(1+k)^r} \frac{\Gamma(B+1+r)}{\Gamma(B+1)r!}$$

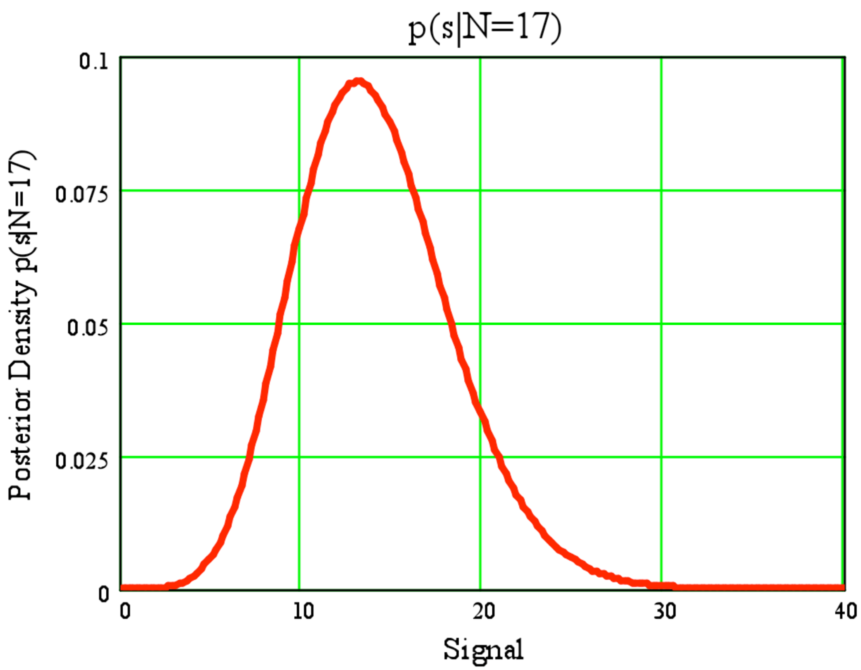
we can compute

$$p(s \mid n, H_1) = \frac{p(n \mid s, H_1)\pi(s \mid H_1)}{\int_0^\infty p(n \mid s, H_1)\pi(s \mid H_1)\, ds}$$

Example 2: Posterior Density

Assuming a ***flat prior*** for the signal $\pi(s|H_1) = constant$, the posterior density is given by

$$p(s \mid n) = \frac{\sum_{r=0}^{n} c_r(k,B)\,\mathrm{Poisson}(n-r \mid s)}{\sum_{r=0}^{n} c_r(k,B)}$$



Example 2: Posterior Density

Next compute the moments of p(s|n) about zero

$$M_m = \int_0^\infty s^m p(s \mid n) ds = \sum_{r=0}^{n} c_r(k,B)(n-r+m)! / (n-r)! / \sum_{r=0}^{n} c_r(k,B)$$

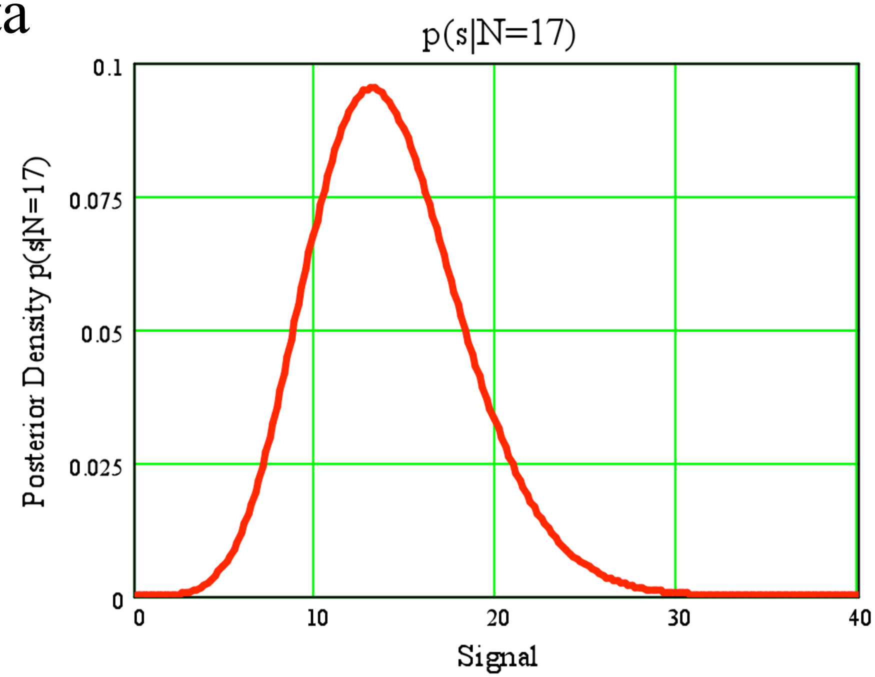
For the top quark discovery data we find:

mean

$M_1 = 14.0$ events

standard deviation

$\sqrt{(M^2 - M_1^2)} = 4.3$ events



Example 3: Evidence

By definition, the evidence for hypothesis H_0 is

$$p(N \mid H_0)$$

and

$$p(N \mid H_1) = \int_0^\infty p(N \mid s, H_1)\pi(s \mid H_1)\, ds$$

for the alternative hypothesis H_1. To proceed, we need to specify a ***proper*** prior for the signal. For simplicity, we shall take the prior to be a delta function at ***s*** = **14** events.

This yields:

$$p(N \mid H_0) = \mathbf{3.86 \times 10^{-6}}$$

$$p(N \mid H_1) = p(N \mid s{=}14, H_1) = \mathbf{9.28 \times 10^{-2}}$$

Example 3: Evidence

Since,

$$p(N \mid H_0) \quad = \mathbf{3.86 \text{ x } 10^{-6}}$$

$$p(N \mid H_1) \quad = \mathbf{9.28 \text{ x } 10^{-2}}$$

we conclude that the hypothesis with ***s*** = 14 events is favored over that with ***s*** = 0 by **24,000** to **1**.

In terms of a measure akin to "n-sigma," this is

$$\kappa = \sqrt{2 \ln\left[p(N \mid H_1) / p(N \mid H_0) \right]} = 4.5$$

which may be compared with the 4.4 σ obtained via the p-value.

Summary

In these examples, we have shown how to

1. Take into account imprecisely known background

2. Estimate the signal and quantify how well this has been done

3. Test the background + signal hypothesis, H_1, against the background-only hypothesis, H_0.

The End

“Have the courage to use your ***own*** understanding!”

Immanuel Kant