PHY 4241/5227 : Advanced Dynamics, Spring 2007

February 9^{th}, 2007
Assignment # 5
(due Friday February 16^{th}, 2007, at the beginning of class)

1. Thanks to **gauge transformations** clearly distinct vector potentials can generate the same exact magnetic field.

1.a) Show that if two vector potentials, $\mathbf{A}_1(\mathbf{x})$ and $\mathbf{A}_2(\mathbf{x})$ differ by a gradient of a scalar function $\Lambda(\mathbf{x})$, then both potentials generate the same exact magnetic field $\mathbf{B}(\mathbf{x})$. That is, if $\mathbf{A}_2(\mathbf{x}) = \mathbf{A}_1(\mathbf{x}) + \nabla\Lambda(\mathbf{x})$ and $\mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A}_1(\mathbf{x})$, then $\mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A}_2(\mathbf{x})$.
Now consider the following three vector potentials with B a constant:

$$\mathbf{A}_1(\mathbf{x}) = B(x\hat{\mathbf{y}} - y\hat{\mathbf{x}})/2 \ , \ \ \mathbf{A}_2(\mathbf{x}) = Bx\hat{\mathbf{y}} \ , \ \ \mathbf{A}_3(\mathbf{x}) = -By\hat{\mathbf{x}} \ ,$$

1.b) Show that all three vector potentials satisfy: $\nabla \times \mathbf{A} = B\hat{\mathbf{z}}$ and $\nabla \cdot \mathbf{A} = 0$.

1.c) Find the three gauge transformations, *i.e.* the three scalar functions $\Lambda_{12}(\mathbf{x})$, $\Lambda_{23}(\mathbf{x})$, and $\Lambda_{31}(\mathbf{x})$, that connect the above gauge-equivalent vector potentials.

2. Consider a particle of mass m and charge q moving under the influence of a constant magnetic field of the form $\mathbf{B}(\mathbf{x}) = B\hat{\mathbf{z}}$.

2.a) Using any of the vector potentials in Problem 1, show that the velocity-dependent potential

$$V(\mathbf{x}, \dot{\mathbf{x}}) = -\frac{q}{c}\dot{\mathbf{x}} \cdot \mathbf{A}(\mathbf{x}) \ ,$$

generates, through the use of the Euler-Lagrange equations of motion, the well known force

$$\mathbf{F} = m\ddot{\mathbf{x}} = \frac{q}{c}\dot{\mathbf{x}} \times \mathbf{B} \ .$$

2.b) Evaluate the Hamiltonian function

$$h(\mathbf{x}, \dot{\mathbf{x}}) = \dot{x}_i p_i - L(\mathbf{x}, \dot{\mathbf{x}}) \ ; \ p_i = \frac{\partial L}{\partial \dot{x}_i} \ ,$$

and show that it is equal to the kinetic energy of the particle. Is h a constant of motion?

2.c) Solve the Euler-Lagrange equations of motion subject to the following initial conditions: $x(0) = \dot{y}(0) = 1$ and $y(0) = z(0) = \dot{x}(0) = \dot{z}(0) = 0$.

3. Derive esplicitely Eqs (10.29) and (10.31) of Griffiths's book from the expressions of the retarded potentials $\Phi(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$ in Eq (10.19) (use the units you prefer).