Main Reference { your book
Bogoliubov & Shirkov, "Introduction to the theory of Quantized Fields" ("Noether's theorem and dynamics invariants"), 1

(Global)

Symmetries and Conservation laws : Noether's theorem

"If the action functional (S) of a classical system of fields $q_i(x)$ is invariant under the action of a continuous group of transformations dependent on a finite number of parameters n, then the system admits n dynamical invariants, i.e. n conserved quantities (over time)".

↳ the group of transformations acts on both (or can act) coordinates and fields simultaneously.

Infinitesimal transformation:

$$x^\mu \to x'^\mu = x^\mu + \delta x^\mu \cong x^\mu + \sum_{k=1}^{n} X^\mu_{(k)} \alpha_k$$

$$q_i(x) \to q'_i(x') = q_i(x) + \delta q_i \cong q_i(x) + \sum_{k=1}^{n} \Phi_{i(k)} \alpha_k$$

$\{\alpha_k\} \to$ parameters — not x-dependent

Under this transformation:

$$\mathcal{L}(q_i(x), \partial_\mu q_i(x), x) \longrightarrow \mathcal{L}'(q'_i(x'), \partial'_\mu q'_i(x'), x').$$

$$\mathcal{L}(x) \qquad\qquad \mathcal{L}'(x')$$

we want to compute:

$$\delta S = \frac{dS}{d\alpha}d\alpha = S(\alpha) - S(0) = \int d^4x' \mathcal{L}'(x') - \int d^4x\, \mathcal{L}(x)$$

$$S(\alpha) = S(0) + \sum_{k=1}^{n} \alpha_k \left.\frac{dS}{d\alpha_k}\right|_{\alpha=0} + O(\alpha^2)$$

extremum: $= 0$

and see the consequences of setting : $\delta S = 0$

First of all: → let's separate the variation due to δx^μ from the variation due to δq_i

$$\mathcal{L}'(x') = \mathcal{L}(x) + \delta\mathcal{L}(x)$$

$$= \mathcal{L}(x) + \bar{\delta}\mathcal{L}(x) + \frac{d\mathcal{L}(x)}{dx^\mu}\delta x^\mu$$

$$\bar{\delta}\mathcal{L}(x) = \frac{\partial \mathcal{L}}{\partial q_i}\bar{\delta} q_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu q_i)}\bar{\delta}(\partial_\mu q_i) \qquad \frac{\partial \mathcal{L}}{\partial q_i}\frac{\partial q_i}{\partial x^\mu} + \frac{\partial \mathcal{L}}{\partial(\partial_\nu q_i)}\frac{\partial(\partial_\nu q_i)}{\partial x^\mu} + \frac{\partial \mathcal{L}}{\partial x^\mu}$$

$\bar{\delta}$ → variation due only to change of q_i not due to δx^μ. (variation in form of q_i, not variation in its argument) ⇒

$$\bar{\delta} q_i = q_i'(x) - q_i(x) = \delta q_i - \partial_\nu q_i \delta x^\nu$$

$$\bar{\delta}(\partial^\mu q_i) = \partial^\mu(\bar{\delta} q_i)$$

$$\bar{\delta}\mathcal{L}(x) = \frac{\partial \mathcal{L}}{\partial q_i}\bar{\delta} q_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu q_i)}\partial_\mu(\bar{\delta} q_i)$$

$$= \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu q_i)}\bar{\delta} q_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu q_i)}\partial_\mu(\bar{\delta} q_i)$$

$$= \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu q_i)}\bar{\delta} q_i\right)$$

So:

$$\mathcal{L}'(x') = \mathcal{L}(x) + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu q_i)}\bar{\delta} q_i\right) + \frac{d\mathcal{L}(x)}{dx^\mu}\delta x^\mu$$

Moreover:

$$d^4x' = dx_0' dx_1' dx_2' dx_3' = |J|\, d^4x =$$

$$= \det\left(\frac{\partial x'^\mu}{\partial x^\nu}\right) d^4x \cong \left(1 + \frac{\partial \delta x^\mu}{\partial x^\mu}\right) d^4x$$

$$\det(1+\epsilon) = 1 + \mathrm{Tr}(\epsilon)$$

So :

$$\delta S = \int d^4x' \mathcal{L}'(x') - \int d^4x \, \mathcal{L}(x)$$

$$= \int d^4x \left[\mathcal{L}(x) + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu q_i)} \bar{\delta} q_i \right) + \frac{d\mathcal{L}(x)}{dx^\mu} \delta x^\mu \right] \left(1 + \frac{\partial \delta x^{\mu'}}{\partial x^{\mu'}} \right)$$

$$- \int d^4x \, \mathcal{L}(x)$$

$$= \int d^4x \left[\cancel{\mathcal{L}(x)} + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu q_i)} \bar{\delta} q_i \right) + \frac{d\mathcal{L}(x)}{dx^\mu} \delta x^\mu + \mathcal{L}(x) \frac{\partial \delta x^\mu}{\partial x^\mu} - \cancel{\mathcal{L}(x)} \right]$$

$$= \int d^4x \left[\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu q_i)} \bar{\delta} q_i \right) + \partial_\mu \left(\mathcal{L}(x) \delta x^\mu \right) \right]$$

we use ∂_μ as d_μ indeed from then on, since there are no more ambiguities.

$$= \int d^4x \, \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu q_i)} \bar{\delta} q_i + \mathcal{L}(x) \delta x^\mu \right]$$

$$= \int d^4x \, \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu q_i)} \left(\delta q_i - \partial_\nu q_i \, \delta x^\nu \right) + \mathcal{L}(x) \delta x^\mu \right]$$

$$= - \sum_{k=1}^{n} \int d^4x \, \partial_\mu \left[- \mathcal{L}(x) X^\mu_{(k)} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu q_i)} \left(\Phi_{i(k)} - \partial_\nu q_i \, X^\nu_{(k)} \right) \right] \alpha_k$$

$$= - \sum_{k=1}^{n} \int d^4x \, \partial_\mu J^\mu_{(k)}(x) \, \alpha_k$$

where :

$$J^{\mu}_{(k)}(x) = -\mathcal{L}(x)\,X^{\mu}_{(k)} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu q_i)}\left(\Phi_{i(k)} - \partial_\nu q_i\, X^{\nu}_{(k)}\right)$$

so, finally :

$$\delta S = 0 \quad \longrightarrow \quad -\sum_{k=1}^{n} \int d^4x\, \partial_\mu J^{\mu}_{(k)}(x)\, \alpha_k = 0$$

since the $\{\alpha_k\}$ are independent :

$$\delta S = 0 \quad \rightarrow \quad \frac{\partial S}{\partial \alpha_k} = 0 \quad \rightarrow \quad \int d^4x\, \partial_\mu J^{\mu}_{(k)}(x) = 0$$

and because of the arbitrariness of the integration region this implies :

$$\partial_\mu J^{\mu}_{(k)}(x) = 0 \qquad k = 1, \dots, n$$

$\hookrightarrow$ <u>n conserved currents</u>

Moreover, for arbitrary space-like 3D surfaces σ_1 and σ_2 we can rewrite :

$$\int d^4x\, \partial_\mu J^{\mu}_{(k)}(x) = \int_{\partial\Omega} dS_\mu\, J^{\mu}_{(k)}(x) = \int_{\sigma_1} dS_\mu\, J^{\mu}_{(k)} - \int_{\sigma_2} dS_\mu\, J^{\mu}_{(k)}$$

and

$$\int_{\sigma} dS_\mu\, J^{\mu}_{(k)}(x) = \text{constant}$$

choose $\sigma(t_0)$; $t_0 = \text{constant}$

$$C_k(t_0) = \int d^3x\, J^{0}_{(k)}(x) = \text{constant} \qquad k = 1, \dots, n$$

$\hookrightarrow$ <u>n conserved charges</u>, "constant of motion" quantities the currents $J^\mu_{(k)}(x)$ are defined up to an arbitrary term of the form:

$$\partial_\nu f^{\mu\nu}_{(k)} \qquad \text{s.t.} \quad f^{\mu\nu}_{(k)} = -f^{\nu\mu}_{(k)}$$

since then:

$$\partial_\mu \partial_\nu f^{\mu\nu}_{(k)} = 0$$

(This goes back to the fact that the Lagrangian is defined up to a total 4-divergence).

1st example : Energy-momentum vector / Tensor

infinitesimal space translations : $x'^\mu = x^\mu + a^\mu$

$$q'_i(x') = q_i(x)$$

such that:

$$\delta x^\mu = \sum_{k=1}^{n} X^\mu_{(k)} \alpha_k \quad \text{with} \quad X^\mu_{(k)} = \delta^\mu_\nu \qquad \alpha_k = a^\nu$$

$$\delta q_i = \sum_{k=1}^{n} \Phi_{i(k)} \alpha_k \quad \text{with} \quad \Phi_{i(k)} = 0$$

the conserved currents are:

$$\underset{\hookrightarrow \nu}{J^\mu_{(k)}}(x) = -\mathcal{L}(x)\,\delta^\mu_\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu q_i)} \partial_\nu q_i$$

$$= -\mathcal{L}(x)\, g^{\mu\nu} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu q_i)} \partial^\nu q_i = T^{\mu\nu}$$

"energy momentum tensor"

and the conserved charges are:

$$P^\mu = \int d^3x\, T^{\mu 0}, \quad \text{s.t.} \quad E = \int d^3x\, T^{00}$$

$$\left(= \frac{\partial \mathcal{L}}{\partial(\partial_0 q_i)} \partial^0 q_i - \mathcal{L} = H \right)$$

this is why we picked the overall $(-)$ sign.

and therefore the other three components of $\vec{P}$

2nd example : Angular Momentum Tensor and Spin Tensor

Lorentz Transformation :
$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$
$$q'_i(x') = L_{ij}(\Lambda) q_j(x)$$
$\downarrow$ suitable representation of the Lorentz group.

For infinitesimal transformations, i.e. in the vicinity of the identity (of the group):

$$x'^{\mu} = x^{\mu} + \sum_{k=1}^{6} \alpha_k \Omega^{\mu}{}_{\nu(k)} x^{\nu}$$

$$q'_i(x') = q_i(x) + \sum_{k=1}^{6} \alpha_k A_{ij(k)} q_j(x)$$

where $\Omega^{\mu}{}_{\nu(k)}$ are the generators of the Lorentz group (6) and α_k six independent parameters.

Consider:
$$\Omega^{\mu}{}_{\nu} = \sum_{k=1}^{6} \alpha_k \Omega^{\mu}{}_{\nu(k)} \qquad \text{(generic element of the group)}$$

then we know that : $\Omega^t g + g\Omega = 0$

$$(\Omega^t)^{\mu}{}_{\ell}\, g^{\ell\nu} + g^{\mu\ell}\, \Omega_{\ell}{}^{\nu} = 0$$

$$g^{\nu\ell}\Omega_{\ell}{}^{\mu} + g^{\mu\ell}\Omega_{\ell}{}^{\nu} = 0 \qquad \Omega^{\nu\mu} + \Omega^{\mu\nu} = 0$$

$$\Omega^{\nu\mu} = -\Omega^{\mu\nu}$$

if we use the same $\Omega^{\mu\nu}$ as infinitesimal parameters of the transformation then :

$$x'^{\mu} = x^{\mu} + \Omega^{\mu\nu} x_{\nu}$$

and to match the

$$x'^{\mu} = x^{\mu} + \sum_k X^{\mu}_k \alpha_k$$ we need to replace $(k) \leftrightarrow [\mu\nu]$ antysimm

ansatz:
$$x'^{\mu} = x^{\mu} + \sum_{\rho<\sigma}{}' X^{\mu}_{[\rho\sigma]}\,\Omega^{\rho\sigma}$$

then:

$$\sum_{\rho<\sigma} X^{\mu}_{[\rho\sigma]}\,\Omega^{\rho\sigma} = \Omega^{\mu\nu}x_{\nu} = \delta^{\mu}_{\rho}\,\Omega^{\rho\sigma}x_{\sigma} =$$
$$= \sum_{\rho<\sigma}\delta^{\mu}_{\rho}\,\Omega^{\rho\sigma}x_{\sigma} + \sum_{\rho>\sigma}\delta^{\mu}_{\rho}\,\Omega^{\rho\sigma}x_{\sigma}$$
$$= \sum_{\rho<\sigma}\delta^{\mu}_{\rho}\,\Omega^{\rho\sigma}x_{\sigma} - \sum_{\rho>\sigma}\delta^{\mu}_{\rho}\,\Omega^{\sigma\rho}x_{\sigma}$$
$$= \sum_{\rho<\sigma}\delta^{\mu}_{\rho}\,\Omega^{\rho\sigma}x_{\sigma} - \sum_{\rho<\sigma}\delta^{\mu}_{\sigma}\,\Omega^{\rho\sigma}x_{\rho}$$
$$= \sum_{\rho<\sigma}\underbrace{\left(\delta^{\mu}_{\rho}x_{\sigma} - \delta^{\mu}_{\sigma}x_{\rho}\right)}_{X^{\mu}_{[\rho\sigma]}}\Omega^{\rho\sigma}$$

For δq_i is just:

$$A_{ij(k)} \longrightarrow A_{ij[\rho\sigma]}$$

then, the conserved currents are:

$$J^{\mu}_{[\rho\sigma]} = -\mathcal{L}(x)\left(\delta^{\mu}_{\rho}x_{\sigma} - \delta^{\mu}_{\sigma}x_{\rho}\right) - \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}q_i)}\Big[A_{ij[\rho\sigma]}\,q_j(x) + $$
$$- \partial_{\nu}q_i\left(\delta^{\nu}_{\rho}x_{\sigma} - \delta^{\nu}_{\sigma}x_{\rho}\right)\Big]$$

$$= \left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}q_i)}\partial_{\rho}q_i - \mathcal{L}(x)\,\delta^{\mu}_{\rho}\right)x_{\sigma} +$$
$$\left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}q_i)}\partial_{\sigma}q_i - \mathcal{L}(x)\,\delta^{\mu}_{\sigma}\right)x_{\rho} +$$
$$- \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}q_i)}A_{ij[\rho\sigma]}\,q_j(x)$$

We recognize in (..) the energy-momentum tensor.
So, the new conserved currents look like:

$$M^{\mu}_{[\ell\sigma]} = \underbrace{T^{\mu}_{\ \ell}\, x_{\sigma} - T^{\mu}_{\ \sigma}\, x_{\ell}}_{\text{orbital angular momentum}} - \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} A_{ij[\ell\sigma]}\, \varphi_j(x)}_{\text{spin angular momentum}}$$

s.t.

$$\partial_\mu M^{\mu}_{[\ell\sigma]}(x) = 0$$

The corresponding conserved charges are:

$$M_{[\ell\sigma]}(t) = \int_{t=\text{const.}} d^3x\ M^{0}_{[\ell\sigma]}(x) \longrightarrow \text{angular momentum tensor}$$

$(T^{\mu}_{\ \ell}\, x_{\sigma} - T^{\mu}_{\ \sigma}\, x_{\ell}) \longrightarrow$ depends only on the coordinates

$S^{\mu}_{[\ell\sigma]} = \left(- \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} A_{ij[\ell\sigma]}\, \varphi_j(x) \right) \longrightarrow$ " " " " internal degrees of freedom (i.e. Lorentz properties of the field φ_i)

ex: scalar field $\rightarrow$ $A_{ij[\ell\sigma]} = 0$ $\rightarrow$ only orbital ang. momentum.

Ex. 3: The same can be repeated for any internal symmetry of the fields $\{\varphi_i(x)\}$, i.e. a symmetry that acts only on the field.
We'll see examples later on.