This problem is taken from *Entropy, Order Parameters, and Complexity* by James P. Sethna and goes under the name of **Quantum dice**. It deals with discrete distributions and serves as a preview/review to Bose and Fermi statistics. Here is how it goes.

You are given several unusual “three-sided” dice which, when rolled, show either one, two, or three spots. There are three games played with these dice: **Distinguishable**, **Bosons**, and **Fermions**. In each turn in these games, the player rolls one die at a time, starting over if required by the rules, until a legal combination occurs. In **Distinguishable**, all rolls are legal. In **Bosons**, a roll is legal only if the new number is larger or equal to the preceding number. In **Fermions**, a roll is legal only if the new number is strictly larger than the preceding number.

1. **[5pt]** Notice that our dice rules are the same that govern the quantum statistics of identical particles. Can you explain why? In other words: what is the role played by the dice and the role played by the sides of the dice?

2. **[5pt]** Build a table of possibilities after rolling two dice for each of the games, i.e. what are the possible legal combinations after rolling two dice when playing **Distinguishable**, **Bosons**, and **Fermions**?

3. **[5pt]** Presume that the dice are fair (i.e. each of the three numbers of dots shows up 1/3 of the time). Among the legal **Bosons** turn rolling two dice, what is the probability \( \rho(4) \) of rolling a 4. Similarly, what is \( \rho(4) \) when playing **Fermions**?

4. **[5pt]** For a legal turn rolling three “three-sided” dice in **Fermions**, what is the probability \( \rho(6) \) of rolling 6?

5. **[10pt]** When rolling \( M \) dice each with \( N \) sides, how many legal turns there are in **Distinguishable**, **Bosons**, **Fermions**.

6. **[10pt]** In a turn of three rolls, what is the factor by which the probability of getting triplets in **Bosons** is enhanced over that in **Distinguishable**? In a turn of \( M \) rolls, what is the enhancement factor for generating an \( M \)-tuple (all rolls having the same number of dots showing)?

   Notice that the states of the dice tend to cluster together in **Bosons**. Examples of real bosons clustering into the same state include Bose condensation and lasers.