Problem 1

Consider the problem of the vibrational modes in a solid satisfying the following dispersion relation,

\[ \omega(k) = A|k|^s \equiv Ak^s , \]

where \( A \) and \( s \) are positive constants, \( \omega \) is the angular frequency, and \( k \) the wave number of the mode. Assume that there are \( N \) atoms in the solid so that the total number of modes is equal to \( 3N \).

1.a) Compute the Debye wave number \( k_D \). Does \( k_D \) depend on the assumed dispersion relation? What about the Debye frequency \( \omega_D \)?

1.b) Show that the specific heat of the solid at low temperatures is proportional to \( T^{3/s} \). Note that while \( s = 1 \) corresponds to the case of elastic waves in a lattice (phonons), \( s = 2 \) applies to spin waves (magnons) propagating in a ferromagnetic system.

1.c) Compute the specific heat of the solid at high temperatures and compare your result to the law of Dulong and Petit (classical result, i.e. \( C_V = 3Nk_B \)).

Problem 2

In the one-dimensional Ising model \( N \) localized spins are fixed to the different sites of an evenly spaced one-dimensional lattice that is placed in a constant magnetic field \( B \). The spins, which are limited to only two values \( (s_i = \pm 1) \), interact with the magnetic field and with each other through a classical spin-spin interaction. The Ising Hamiltonian for such a system is given by,

\[ H = -\mu B \sum_{i=1}^{N} \frac{1}{2}(s_i + s_{i+1}) - J \sum_{i=1}^{N} s_is_{i+1} . \]

Here \( \mu \) denotes the strength of the spin coupling to the external magnetic field and \( J > 0 \) is the ferromagnetic coupling constant. Note that the lattice is assumed to be periodic so that the \((N + 1)\)th spin is equal to the first one. i.e. \( s_{N+1} \equiv s_1 \).
2.a) Show that the partition function of the system may be written as the trace of the $N$th power of a $(2 \times 2)$ matrix. That is,

$$Z(N, T, B) = \text{Tr} \left( \hat{Z}^N \right),$$

where the matrix elements of the $(2 \times 2)$ transfer matrix $\hat{Z}$ are given by,

$$\langle s_1 | \hat{Z} | s_2 \rangle \equiv \exp \left( \beta \mu B (s_1 + s_2)/2 + \beta J s_1 s_2 \right).$$

2.b) Use the fact that the trace of a matrix is independent of the choice of basis to show that $Z(N, T, B)$ may be written as,

$$Z(N, T, B) = \lambda_+^N + \lambda_-^N,$$

where $\lambda_+$ and $\lambda_-$ are the larger and smaller eigenvalues of $\hat{Z}$, respectively.

2.c) Show that in the thermodynamic ($N \to \infty$) limit the Helmholtz free energy of the system may be written as,

$$\frac{1}{N} F(N, T, B) = -k_B T \ln \lambda_+ = -J - k_B T \ln \left[ \cosh(\beta \mu B) + \sqrt{\sinh^2(\beta \mu B) + e^{-4\beta J}} \right].$$

2.d) Compute the average magnetization $M$ of the system, i.e. the number of spins up ($N_+$) relative to the number of spins down ($N_-$), using the relation,

$$M = - \left( \frac{\partial F}{\partial B} \right)_{N, T},$$

and study its leading behavior in the limit of $\beta \mu B \ll 1$ and $\beta \mu B \gg 1$. Conclude that there is no spontaneous magnetization by showing that $M \to 0$ as $B \to 0$ for all temperatures.