PHY 5524: Statistical Mechanics

March 21^{st}, 2012

Assignment # 11

(Due Wednesday March 28^{th}, 2012)

Problem 1

An ideal Bose gas that is in contact with a thermal/particle reservoir at a temperature T and chemical potential μ obeys the following dispersion relation,

$$\epsilon(\mathbf{p}) = A|\mathbf{p}|^\alpha \equiv Ap^\alpha \ ,$$

where A and α are positive constants. In answering the following questions you might ignore the degeneracy factor (*i.e.* assume $g = 1$).

1.a) Derive an integral expression for the number density (N/L^D) of the system, by using the following expression for the volume element of phase space in D-dimensions,

$$\frac{d^D p \, d^D x}{h^D} = \Omega_D \frac{L^D \, p^{D-1} \, dp}{h^D} \ .$$

1.b) Keeping the number density fixed, discuss the limit $\mu \to 0$ (or $z = 1$) and determine the onset (or lack) of Bose condensation as a function of α and D. For those cases in which the system condenses, obtain the critical temperature T_c and determine the fraction of particles in the condensate for $T < T_c$. **Note that you do not need to explicitly evaluate any integrals**.

Problem 2

An ideal Bose gas that is in contact with a thermal/particle reservoir at temperature T and chemical potential μ obeys the following non-relativistic dispersion relation,

$$\epsilon(\mathbf{p}) = \frac{|\mathbf{p}|^2}{2m} = \frac{p^2}{2m} \ .$$

2.a) Compute the energy E as a function of μ (or z), T, and V.

2.b) Obtain an exact expression for the specific heat of the system in terms of the Bose functions $g_\nu(z)$. **Do not** attempt to evaluate the integrals exactly.

2.c) Using Maple, Mathematica, or any tool of your choice, plot the specific heat of the system as a function of T/T_c.

2.d) Show that the specific heat of the system vanishes as $T^{3/2}$.

2.e) Find an explicit value for the specific heat of the system at $T = T_c$ in terms of ratios of two ζ functions and show that it exceeds the classical value.

2.f) Finally show that the specif heat of the system approaches its classical value for $T \gg T_c$.

Problem 3

This problem will not be graded. It is just for you to read more about the experimental evidence for Bose-Einstein condensation and understand it. Attached to this homework is the original paper that describes how Anderson, Ensher, Matthews, Wieman, and Cornell (in 1995) were able to get a dilute gas of rubidium-87 to Bose-condense. We will discuss some points of statistical mechanical interest during problem session on Monday, March 28th. It would be very useful if you could read it before then and try to estimate some of the quantities described in the following itemized points (from J. P. Sethna's book *Entropy, order parameters, and complexity*, Oxford University Press).

Also, I suggest you visit the official BEC site of the University of Colorado at Boulder (`http://jila.colorado.edu/bec/`), where you will be able to learn everything about this *cool* phenomenon!

3.a) *Is rubidium-87 (37 protons and electrons, 50 neutrons) a boson or a fermion?*

3.b) *At their quoted maximum number density of* $2.5 \times 10^{12}/\mathrm{cm}^3$*, at what temperature* T_c^{predict} *do you expect the onset of Bose condensation in free space? They claim that they found Bose condensation starting at a temperature of* $T_c^{\mathrm{measured}} = 170$ *nK. Is that above or below your estimate?* (Useful constants: $h = 6.6262 \times 10^{-27}$ erg s, $m_n \approx m_p = 1.6726 \times 10^{-24}$ g, $k_B = 1.3807 \times 10^{-16}$ erg/K.)

3.c) The trap had an effective potential energy that was harmonic in the three dimensions, but anisotropic with cylindrical symmetry. The frequency along the cylindrical axis was $f_0 = 120$ Hz, so $\omega_0 \approx 750$ Hz, and the two other frequencies were smaller by a factor $\sqrt{8}$, i.e. $\omega_1 \approx 265$ Hz. The Bose condensation was observed by abruptly removing the trap potential and letting the gas atoms spread out; the spreading cloud was imaged 60 ms later by shining a laser on them and using a CCD to image the shadow (see Fig. 2 in the paper or the beautiful 3D rendering on the webpage cited above). For your convenience, the ground state of a particle of mass m in a one-dimensional harmonic oscillator potential with frequency ω is $\psi_0(x) = (m\omega/\pi\hbar)^{1/4} e^{-m\omega x^2/(2\hbar)}$, and the momentum-space wavefunction is $\tilde{\psi}_0(p) = (1/(\pi\hbar m\omega))^{1/4} e^{-p^2/(2m\hbar\omega)}$. The 3D wavefunctions are then the product of the corresponding 1D wavefunctions along the three axes.

Will the momentum distribution be broader along the high-frequency axis (ω_0) *or one of the low-frequency axes* (ω_1)*?* Assume that you may ignore the small width in the initial position distribution, and that the positions in Fig. 2 reflect the velocity distribution times the time elapsed. *Which axis, x or y in Fig. 2, corresponds to the high-frequency cylinder axis? What anisotropy does one expect in the momentum distribution at high temperatures (classical statistical mechanics)?*

3.d) Their Bose condensation is not in free space; the atoms are in a harmonic oscillator potential. In the calculation in free space, we approximated the quantum states as a continuum density of states $g(E)$. That is only sensible if $k_B T$ is large compared to the level spacing near the ground state. *Compare $\hbar\omega$ to $k_B T$ at the Bose condensation point T_c^{measured} in their experiment*

3.e) For bosons on a one-dimensional harmonic oscillator potential of frequency ω_0, it is clear that $g(E) = 1/(\hbar\omega_0)$; the number of states in a small range ΔE is the number of $\hbar\omega_0$s it contains. *Compute the density of single-particle eigenstates,*

$$g(E) = \int_0^\infty d\epsilon_1\, d\epsilon_2\, d\epsilon_3\, g_1(\epsilon_1)\, g_2(\epsilon_2)\, g_3(\epsilon_3)\delta(E - \epsilon_1 - \epsilon_2 - \epsilon_3) \ ,$$

for a three-dimensional harmonic oscillator, with one frequency ω_0 and two frequencies ω_1.

3.f) Their experiment has $N = 2 \times 10^4$ atoms in the trap as it condenses. *By working in analogy with the calculation in free space, find the maximum number of atoms that can occupy the three-dimensional harmonic oscillator potential in part (**3.e**) without Bose condensation at temperature T. (You will need to know that $\int_0^\infty x^2/(e^x - 1)\, dx = 2\,\zeta(3) = 2.40411$.) According to your calculation, at what temperature T_c^{HO} should the real experimental trap have Bose condensed?*