

PHY 5524: Statistical Mechanics

March 28th, 2012

Assignment # 12

(Due Wednesday April 4th, 2012)

Problem 1

An ideal Bose gas contained in a box of fixed volume V consists of N noninteracting bosons of mass M each of which possesses an internal degree of freedom which can be described by assuming that the bosons are *two-level* systems. Bosons (with a fixed momentum \mathbf{p}) in the ground state have energy $E_0 = p^2/2M$, while bosons in the excited state have energy $E_1 = p^2/2M + \Delta$, where $\Delta > 0$ is the excitation energy. Assume that $\Delta \gg k_B T$.

- 1.a) Compute the Bose-Einstein condensation temperature T_c for this gas of two-level bosons.
- 1.b) Obtain an expression for the amount by which the condensation temperature is raised or lowered due to the existence of the internal degree of freedom.
- 1.c) For the temperatures below T_c , obtain an expression for the condensate fraction, *i.e.* the fraction of bosons in their ground state which occupy the zero-momentum state.

Problem 2

Consider a gas of photons ($\mu \equiv 0$) in a cavity of volume V in equilibrium with a thermal reservoir at a temperature T .

- 2.a) Compute the energy density (E/V), the radiation pressure (P), and the entropy (S) of the system. Verify that $P = (E/V)/3$.
- 2.b) Now assume that the volume of the cavity increases *isentropically* (at constant entropy). Show that during such an isentropic expansion the product VT^3 remains constant.
- 2.c) When the universe cooled to about a temperature of $T = 3,000$ K, the electrons and the protons combined to form neutral hydrogen atoms. After this *recombination era*, photons were able to travel through the universe relatively unimpeded, *i.e.* the universe became transparent. What was the radius of the universe then relative to what it is now? (Assume that the present temperature of the cosmic black-body radiation is $T = 3$ K).

Problem 3

(from J. P. Sethna's book *Entropy, order parameters, and complexity*, Oxford University Press).

The experiment the Planck was studying did not directly measure the energy density per unit frequency inside a box. It measured the energy radiating out of a small hole, of area A . Assume that the hole is on the upper part of the cavity, perpendicular to the z axis.

Indicate with $v_z = c \cos \theta$ the vertical component of the velocity of each photon, where θ is the angle between the photon velocity and the vertical. The photon distribution just inside the boundary of the hole is depleted of photons with $v_z < 0$ (very few photons come into the hole from the outside), but it (almost) unaffected by the presence of the hole for photons with $v_z > 0$.

- 3.a) Show that the probability density $\rho(v_z)$ for a particular photon to have velocity v_z is independent of v_z .
- 3.b) An upper bound on the energy emitted from a hole of area A is given by the energy in the box as a whole times the fraction $A c dt/V$ of the volume within $c dt$ of the hole. Show that the actual energy emitted is $1/4$ of this upper bound.
- 3.c) Can you explain why is it called *black-body radiation*?
- 3.d) How would you expect the power per unit area emitted in equilibrium at temperature T by a colored body (not black) or a white-body? In other words, write,

$$P_{\text{colored}}(\omega, \theta, T) = a(x, y) P_{\text{black}}(\omega, \theta, T)$$

and explain what should be the role played by the function $a(x, y)$ and what do you expect the variables x and y to be.

- 3.e) Finally, calculate the total power per unit area emitted by a black body at temperature T (also known as *Stefan-Boltzmann law*).