PHY 5524: Statistical Mechanics

March 28^{th}, 2012
Assignment # 12
(Due Wednesday April 4^{th}, 2012)

Problem 1

An ideal Bose gas contained in a box of fixed volume V consists of N noninteracting bosons of mass M each of which possesses an internal degree of freedom which can be described by assuming that the bosons are *two-level* systems. Bosons (with a fixed momentum $\mathbf{p}$) in the ground state have energy $E_0 = p^2/2M$, while bosons in the excited state have energy $E_1 = p^2/2M + \Delta$, where $\Delta > 0$ is the excitation energy. Assume that $\Delta \gg k_B T$.

1.a) Compute the Bose-Einstein condensation temperature T_c for this gas of two-level bosons.

1.b) Obtain an expression for the amount by which the condensation temperature is raised or lowered due to the existence of the internal degree of freedom.

1.c) For the temperatures below T_c, obtain an expression for the condensate fraction, *i.e.* the fraction of bosons in their ground state which occupy the zero-momentum state.

Problem 2

Consider a gas of photons ($\mu \equiv 0$) in a cavity of volume V in equilibrium with a thermal reservoir at a temperature T.

2.a) Compute the energy density (E/V), the radiation pressure (P), and the entropy (S) of the system. Verify that $P = (E/V)/3$.

2.b) Now assume that the volume of the cavity increases *isentropically* (at constant entropy). Show that during such an isentropic expansion the product VT^3 remains constant.

2.c) When the universe cooled to about a temperature of $T = 3,000$ K, the electrons and the protons combined to form neutral hydrogen atoms. After this *recombination era*, photons were able to travel through the universe relatively unimpeded, *i.e.* the universe became transparent. What was the radius of the universe then relative to what it is now? (Assume that the present temperature of the cosmic black-body radiation is $T = 3$ K).

Problem 3

(from J. P. Sethna's book *Entropy, order parameters, and complexity*, Oxford University Press).

The experiment the Planck was studying did not directly measure the energy density per unit frequency inside a box. It measured the energy radiating out of a small hole, of area A. Assume that the hole is on the upper part of the cavity, perpendicular to the z axis.

Indicate with $v_z = c\cos\theta$ the vertical component of the velocity of each photon, where θ is the angle between the photon velocity and the vertical. The photon distribution just inside the boundary of the hole is depleted of photons with $v_z < 0$ (very few photons come into the hole from the outside), but it (almost) unaffected by the presence of the hole for photons with $v_z > 0$.

3.a) Show that the probability density $\rho(v_z)$ for a particular photon to have velocity v_z is independent of v_z.

3.b) An upper bound on the energy emitted from a hole of area A is given by the energy in the box as a whole times the fraction $A\,c\,dt/V$ of the volume within $c\,dt$ of the hole. Show that the actual energy emitted is $1/4$ of this upper bound.

3.c) Can you explain why is it called *black-body radiation*?

3.d) How would you expect the power per unit area emitted in equilibrium at temperature T by a colored body (not black) or a white-body? In other words, write,

$$P_{\text{colored}}(\omega, \theta, T) = a(x, y) P_{\text{black}}(\omega, \theta, T)$$

and explain what should be the role played by the function $a(x, y)$ and what do you expect the variables x and y to be.

3.e) Finally, calculate the total power per unit area emitted by a black body at temperature T (also known as *Stefan-Boltzmann law*).