PHY 5524: Statistical Mechanics

April 4^{th}, 2012

Assignment # 13

(Due Wednesday April 11^{th}, 2012)

Problem 1

Consider the problem of the vibrational modes in a solid satisfying the following dispersion relation,

$$\omega(\mathbf{k}) = A|\mathbf{k}|^s \equiv Ak^s \quad ,$$

where A and s are positive constants, ω is the angular frequency, and k the wave number of the mode. Assume that there are N atoms in the solid so that the total number of modes is equal to $3N$.

1.a) Compute the Debey wave number k_D. Does k_D depend on the assumed dispersion relation? What about the Debey frequency ω_D?

1.b) Show that the specific heat of the solid at *low temperatures* is proportional to $T^{3/s}$. Note that while $s = 1$ corresponds to the case of elastic waves in a lattice (*phonons*), $s = 2$ applies to spin waves (*magnons*) propagating in a ferromagnetic system.

1.c) Compute the specific heat of the solid at *high temperatures* and compare your result to the law of Dulong and Petit (classical result, i.e. $C_V = 3Nk_B$).

Problem 2

In the one-dimensional **Ising model** N localized spins are fixed to the different sites of an evenly spaced one-dimensional lattice that is placed in a constant magnetic field B. The spins, which are limited to only two values ($s_i = \pm 1$), interact with the magnetic field and with each other through a classical spin-spin interaction. The Ising Hamiltonian for such a system is given by,

$$H = -\mu B \sum_{i=1}^{N} \frac{1}{2}(s_i + s_{i+1}) - J \sum_{i=1}^{N} s_i s_{i+1} \quad .$$

Here μ denotes the strength of the spin coupling to the external magnetic field and $J > 0$ is the ferromagnetic coupling constant. Note that the lattice is assumed to be periodic so that the $(N+1)$th spin is equal to the first one. i.e. $s_{N+1} \equiv s_1$.

2.a) Show that the partition function of the system may be written as the trace of the Nth power of a (2×2) matrix. That is,

$$Z(N, T, B) = \mathrm{Tr}\left(\hat{\mathcal{Z}}^N\right) \ ,$$

where the matrix elements of the (2×2) *transfer matrix* $\hat{\mathcal{Z}}$ are given by,

$$\langle s_1|\hat{\mathcal{Z}}|s_2\rangle \equiv \exp\left(\beta\mu B(s_1 + s_2)/2 + \beta J s_1 s_2\right) \ .$$

2.b) Use the fact that the trace of a matrix is independent of the choice of basis to show that $Z(N, T, B)$ may be written as,

$$Z(N, T, B) = \lambda_+^N + \lambda_-^N \ ,$$

where λ_+ and λ_- are the larger and smaller eigenvalues of $\hat{\mathcal{Z}}$, respectively.

2.c) Show that in the thermodynamic ($N \to \infty$) limit the Helmoltz free energy of the system may be written as,

$$\frac{1}{N}F(N, T, B) = -k_B T \ln \lambda_+ = -J - k_B T \ln\left[\cosh(\beta\mu B) + \sqrt{\sinh^2(\beta\mu B) + e^{-4\beta J}}\right] \ .$$

2.d) Compute the average magnetization M of the system, i.e. the number of spins *up* (N_+) relative to the number of spins *down* (N_-), using the relation,

$$M = -\left(\frac{\partial F}{\partial B}\right)_{N,T} \ ,$$

and study its leading behavior in the limit of $\beta\mu B \ll 1$ and $\beta\mu B \gg 1$. Conclude that there is no spontaneous magnetization by showing that $M \to 0$ as $B \to 0$ for all temperatures.