PHY 5524: Statistical Mechanics

February 29^{th}, 2012
Assignment # 9
(Due Wednesday March 14^{th}, 2012)

Problem 1

A degenerate electron gas of N particles is confined to a volume V that is in contact with a thermal bath at *zero temperature* and placed in a constant magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$. The density ($n = N/V$) of the system is low enough so that you can safely assume a non-relativistic dispersion relation. That is,

$$\epsilon_{\pm}(\mathbf{p}) = \frac{p^2}{2m} \pm \mu B \ ,$$

where m is the electron mass, $p = |\mathbf{p}|$ is the magnitude of the momentum, and μ is the magnetic moment of the electron.

1.a) Compute the chemical potentials (μ_+ and μ_-) for each of the two (spin up and spin down) electron populations in terms of their respective number densities n_+ and n_-.

1.b) Compute the energy per particle (E/N) of the system in terms of the fraction of the spin populations n_+ and n_-.

1.c) Determine the equilibrium spin populations by minimizing the energy per particle of the system. Note that the spin populations are not independent, i.e. $n_+ + n_- = n$.

1.d) Show that the minimization condition of part (**1.c**) is equivalent to the condition of chemical equilibrium, i.e. $\mu_+ = \mu_-$.

1.e) Compute the electron polarization P defined as

$$P = \frac{n_- - n_+}{n_- + n_+} \ .$$

Problem 2

Consider a zero temperature electron gas at such a high electronic density ($n = N/V$) that it is justified to treat the electrons as a degenerate gas of ultra-relativistic particles satisfying the following dispersion relation:

$$\epsilon(p) = pc \ .$$

2.a) Compute the Fermi momentum of the electron gas. Does the Fermi momentum depend on the dispersion relation?

2.b) Compute the energy-per-particle of the electron gas.

2.c) Compute the pressure of the electron gas.

2.d) Show that the specific heat at constant volume vanishes as $C_v = \gamma T$ and compute explicitly the value of γ.

Hint: You may find the following *Sommerfeld expansion* useful:

$$\int_0^\infty d\epsilon \, \frac{f(\epsilon)}{e^{(\epsilon-\mu)/k_B T} + 1} \simeq \int_0^\mu d\epsilon \, f(\epsilon) + \frac{\pi^2}{6} f'(\mu)(k_B T)^2 \ .$$

Problem 3

In a white-dwarf star nucleons (protons and neutrons) provide most of the mass of the star while electrons most of the pressure against gravitational collapse. As a result of the high electronic density, it is justified to treat the electrons as a *degenerate* gas of fermions at zero temperature. Yet, these electrons are neither non-relativistic nor ultra-relativistic. Thus, the correct treatment must involve a relativistic dispersion relation of the following form:

$$\epsilon(p) = \sqrt{(pc)^2 + (mc^2)^2} \ .$$

3.a) Compute the Fermi momentum of the electron gas. Does the Fermi momentum depend on the dispersion relation?

3.b) Compute the energy-per-particle of the electron gas.

3.c) Compute the pressure of the electron gas.

3.d) Find the energy-per-particle and the pressure in the non-relativistic limit ($p_F c \ll mc^2$) limit.

3.e) Verify that both the energy-per-particle and the pressure have the correct behavior (as obtained in Problem 2) in the ultra-relativistic ($p_F c \gg mc^2$) limit.