

Name: \_\_\_\_\_

PHY 5524: Statistical Mechanics, Spring 2013

April 1<sup>st</sup>, 2013

Midterm Exam # 2, Solutions

Always remember to write full work for what you do. This will help your grade in case of incomplete or wrong answers. Also, no credit will be given for an answer, even if correct, if you give no justification for it.

Write your final answers on the sheets provided. You may separate them as long as you put your name on each of them. We will staple them when you hand them in. Ask if you need extra sheets, they will be provided. Remember to put your name on each of them and add them after the problem they refer to.

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### Part 1

Answer in one or two sentences the following questions.

- (1.a) Write (in terms of the temperature  $T$  and the chemical potential  $\mu$ ) the average occupation number of a single-particle state of energy  $\epsilon$  for a system of bosons. Write the same for a system of fermions.

The average occupation number for a single-particle state of energy  $\epsilon$  is,

$$\begin{aligned} n(\epsilon) &= \frac{1}{e^{\beta(\epsilon-\mu)} - 1} \quad \text{for bosons ,} \\ n(\epsilon) &= \frac{1}{e^{\beta(\epsilon-\mu)} + 1} \quad \text{for fermions .} \end{aligned}$$

- (1.b) Explain briefly the meaning of the chemical potential. What is the range of values allowed for the chemical potential of a fermionic system? What is the range of values if the system is made of bosons?

From a thermodynamical point of view, the basic relation,

$$dE = T dS - P dV + \mu dN$$

defines  $\mu$  as,

$$\mu = \left( \frac{\partial E}{\partial N} \right)_{S,V} ,$$

i.e. as the energy cost of adding a particle at constant entropy (and volume).

At the same time, in statistical mechanics we have seen the chemical potential arising as a Lagrange multiplier introduced to satisfy the constraint of a fixed number of particles, when maximizing the number of microstates available to the system, i.e. the entropy of the system. Since the statistical description of the microscopic degrees of freedom of a system is related to its macroscopic thermodynamics properties, the two definitions are equivalent and complementary in a thorough understanding of thermo-statistic properties of physical systems.

For fermions, since the denominator of  $n(\epsilon)$  never vanishes, the chemical potential can assume any values, i.e.  $-\infty < \mu < \infty$ . On the other hand, for bosons, we need to impose that the denominator in  $n(\epsilon)$  be positive, and, as a consequence, the chemical potential is constrained to be strictly negative, i.e.  $-\infty < \mu < \epsilon_0$ , where  $\epsilon_0$  is the minimal single-particle energy of the system, conventionally chosen to be zero ( $\epsilon_0 = 0$ ).

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## Part 2

Consider a degenerate ultra-relativistic ( $mc^2 \ll cp_F$ ) gas of  $N$  non-interacting electrons in a volume  $V$ . In this limit, the energy of an electron is related to its momentum by  $\epsilon(\mathbf{p}) = c|\mathbf{p}| \equiv cp$ .

- (2.a) Find the chemical potential  $\mu$  and the Fermi momentum  $p_F$  at zero temperature as a function of  $N$  and  $V$ .

The chemical potential at  $T = 0$  is  $\mu = \epsilon_F = cp_F$  where  $p_F$  is obtained as usual from a counting argument (taking  $g = 2$ ), i.e.

$$N = g \frac{V}{h^3} \int d^3p \theta(p_F - |\mathbf{p}|) = \frac{4\pi gV}{h^3} \int_0^{p_F} p^2 dp = \frac{8\pi V}{3h^3} p_F^3 ,$$

such that,

$$p_F = \left( \frac{3N}{8\pi V} \right)^{1/3} h ,$$

and,

$$\mu(T = 0) = \epsilon_F = cp_F = \left( \frac{3N}{8\pi V} \right)^{1/3} hc .$$

- (2.b) Find the total energy  $E$  of the system at zero temperature as a function of  $N$  and  $V$ .

The total energy at zero temperature is,

$$E = \frac{8\pi V}{h^3} \int_0^{p_F} (cp) p^2 dp = \frac{8\pi V}{h^3} \frac{cp_F^4}{4} = \frac{3}{4} N cp_F ,$$

and the energy per particle is,

$$\frac{E}{N} = \frac{3}{4} cp_F .$$

- (2.c) Calculate the pressure  $P$  at zero temperature and show that  $PV = E/3$ . (**Hint:** use  $P = -\partial E/\partial V$  for a  $T = 0$  Fermi gas).

As suggested we use,

$$P = - \left( \frac{\partial E}{\partial V} \right)_N = - \left( \frac{\partial E}{\partial p_F} \right) \left( \frac{\partial p_F}{\partial V} \right) = \frac{p_F}{3V} \left( \frac{3}{4} N c \right) = \frac{1}{4} N \frac{cp_F}{V} ,$$

and we find that,

$$PV = \frac{1}{4} N cp_F = \frac{1}{3} \left( \frac{3}{4} N cp_F \right) = \frac{E}{3} .$$

- (2.d) Find the expression for the *grand canonical potential*  $Q$  at **finite** temperature. Express your result as an integral over energy, function of  $T$ ,  $V$ , and  $\mu$ . You do not have to evaluate the integral.

The *grand canonical potential* at **finite** temperature can be calculated as,

$$\begin{aligned} Q(T, V, \mu) &= -k_B T \ln \mathcal{Z}(T, V, \mu) \\ &= -k_B T \frac{8\pi V}{h^3} \int_0^\infty p^2 \ln[1 + ze^{-\beta\epsilon(p)}] dp \\ &= -k_B T \frac{8\pi V}{(ch)^3} \int_0^\infty \epsilon^2 \ln[1 + ze^{-\beta\epsilon}] d\epsilon, \end{aligned}$$

where, integrating by parts we get,

$$\begin{aligned} Q(T, V, \mu) &= -k_B T \frac{8\pi V}{(ch)^3} \left\{ \frac{\epsilon^3}{3} \ln(1 + ze^{-\beta\epsilon}) \Big|_0^\infty + \beta \int_0^\infty \frac{\epsilon^3}{z^{-1}e^{\beta\epsilon} + 1} d\epsilon \right\} \\ &= -\frac{1}{3} \frac{8\pi V}{(hc)^3} \int_0^\infty \frac{\epsilon^3 d\epsilon}{z^{-1}e^{\beta\epsilon} + 1}. \end{aligned}$$

- (2.e) Find an expression for the total energy  $E$  of the system at **finite** temperature. Express your result as an integral over the energy, function of  $T$ ,  $V$ , and  $\mu$ . You do not have to evaluate the integral.

The total energy  $E$  at **finite** temperature can be calculated as,

$$E = \frac{8\pi V}{h^3} \int_0^\infty \epsilon(p) \frac{p^2 dp}{e^{\beta(\epsilon-\mu)} + 1} = \frac{8\pi V}{(hc)^3} \int_0^\infty \frac{\epsilon^3 d\epsilon}{e^{\beta(\epsilon-\mu)} + 1}.$$

Notice that the same result could be obtained as,

$$E = - \left( \frac{\partial \ln \mathcal{Z}}{\partial \beta} \right)_{z, X},$$

where  $z$  and  $\beta$  are treated as independent variables.

- (2.f) Show that the relation  $PV = E/3$  also holds at **finite** temperature. (**Hint**: integrate your expression for  $Q$  by parts).

Since  $Q = -PV$  we obtain that,

$$PV = \frac{1}{3} \frac{8\pi V}{(hc)^3} \int_0^\infty \frac{\epsilon^3 d\epsilon}{z^{-1}e^{\beta\epsilon} + 1} = \frac{E}{3}.$$