Title: Chiral Anomalies and Their Cancellation, Particularly In The S.M.

Purpose: • Characterize anomalies

• Discuss their significance, importance, (e.g. what they mean for a field theory.)

• Demonstrate the occurrence of the chiral anomaly for a simple theory.

• Demonstrate the occurence and cancellation in the S.M.

• Emphasize the general requirements for chiral anomaly cancellation for other gauge groups (beyond S.M.)

Some useful notes:

Chirality:

$$\psi_{\text{Dirac}} = \begin{pmatrix} \psi_{L,\text{weyl}} \\ \psi_{R,\text{weyl}} \end{pmatrix}$$

$$P_L = \frac{1-\gamma^5}{2}, \quad P_R = \frac{1+\gamma^5}{2}$$

$$P_L + P_R = \mathbb{1}$$

$$(P_L)^2 = P_L, \quad (P_R)^2 = P_R \qquad \text{bc } (\gamma^5)^2 = \mathbb{1}$$

$$P_L \gamma^\mu = \gamma^\mu P_R \qquad \text{bc } \{\gamma^5, \gamma^\mu\} = 0$$

$$P_L \psi_{\text{Dirac}} = \begin{pmatrix} \psi_{L,\text{weyl}} \\ 0 \end{pmatrix}, \quad P_R \psi_{\text{Dirac}} = \begin{pmatrix} 0 \\ \psi_{R,\text{weyl}} \end{pmatrix}$$

$$\gamma^{5\dagger} = \gamma^5 \equiv \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \quad \text{(Weyl Basis)}$$

Symmetries $\longleftrightarrow$ current conservation:

Noether's Theorem - A conserved current is associated with each generator of a continuous symmetry.

Consider a generic continuous field ϕ which transforms infinitesimly as

$$\phi(x) \longrightarrow \phi'(x) = \phi(x) + \underset{\text{infinitesimal}}{\alpha}\, \underbrace{\Delta\phi(x)}_{\text{deformation of field}}$$

Note: The Euler-Lagrange e.o.m.,

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0 ,$$

are left invariant if the action (modulus a surface term), or equivalently if the Lagrangian up to a 4-divergence, is left invariant, e.g.

$$\mathcal{L}(x) \longrightarrow \mathcal{L}'(x) = \mathcal{L}(x) + \alpha\,\partial_\mu J^\mu(x)$$

Varying the fields, we get

$$\alpha\Delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}(\alpha\Delta\phi) + \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\partial_\mu(\alpha\Delta\phi)$$

$$\overset{\text{int. by parts}}{=} \alpha\,\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right) + \alpha\underbrace{\left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\right]}_{=0 \quad \text{E.L. e.o.m.}}\Delta\phi$$

$$\Rightarrow \partial_\mu j^\mu(x) = 0 \quad \text{for } j^\mu(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - J^\mu$$

Consider the theory of a single massless fermion.

$$\mathcal{L} = \bar{\psi} i \gamma^{\mu} \partial_{\mu} \psi$$

Note: $\mathcal{L}$ is invariant under global transformations,

$$\psi' \to e^{i\theta}\psi \quad \text{and} \quad \psi' \to e^{i\theta\gamma^5}\psi.$$

$$\mathcal{L} \to \mathcal{L}' = \bar{\psi}' i \gamma^{\mu} \partial_{\mu} \psi'$$

(i) $$\bar{\psi}' = \psi'^{+}\gamma^0 = \psi^{+} e^{-i\theta}\gamma^0 = \bar{\psi} e^{-i\theta}$$

$$\partial_{\mu}\psi' = i \underbrace{\partial_{\mu}\theta}_{=0} e^{i\theta}\psi + e^{i\theta}\partial_{\mu}\psi$$

$$\Rightarrow \mathcal{L}' = \bar{\psi} e^{-i\theta} i \gamma^{\mu} e^{i\theta} \partial_{\mu}\psi = \bar{\psi} i \gamma^{\mu} \partial_{\mu}\psi = \mathcal{L}$$

(ii) $$\bar{\psi}' = \psi'^{+}\gamma^0 = \psi^{+} e^{-i\theta\gamma^{5+}}\gamma^0 = \psi^{+} e^{-i\theta\gamma^5}\gamma^0$$

$$= \psi^{+}\gamma^0 e^{+i\theta\gamma^5} = \bar{\psi} e^{+i\theta\gamma^5}$$

$$\partial_{\mu}\psi' = i \underbrace{\partial_{\mu}\theta}_{=0} \gamma^5 e^{i\theta\gamma^5}\psi + e^{i\theta\gamma^5}\partial_{\mu}\psi$$

$$\Rightarrow \mathcal{L}' = \bar{\psi} e^{+i\theta\gamma^5} \gamma^{\mu} e^{i\theta\gamma^5} \partial_{\mu}\psi = \bar{\psi} \gamma^{\mu} e^{-i\theta\gamma^5} e^{i\theta\gamma^5} \partial_{\mu}\psi$$

$$= \mathcal{L}$$

where $\{\gamma^5, \gamma^{\mu}\} = 0$

<u>Check</u> Could check that

$$\mathcal{L} \to \mathcal{L}' = \bar{\psi}' i \gamma^\mu \partial_\mu \psi'$$

or equivalently (by Noether's Theorem) check,

$$\partial_\mu j^\mu = 0 \quad \text{and} \quad \partial_\mu j^{\mu 5} = 0 .$$

$$\Rightarrow \partial_\mu j^\mu = \underbrace{(\partial_\mu \bar{\psi})}_{=0} \gamma^\mu \psi + \bar{\psi} \gamma^\mu \underbrace{\partial_\mu \psi}_{=0} = 0 \quad \checkmark$$

$$\partial_\mu j^{\mu 5} = \underbrace{(\partial_\mu \bar{\psi})}_{=0} \gamma^\mu \gamma^5 \psi + \bar{\psi} \gamma^\mu \gamma^5 \underbrace{\partial_\mu \psi}_{=0} = 0 \quad \checkmark$$

Note: If $\mathcal{L} = \mathcal{L}_{Dirac} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi = 0$

the E.L. e.o.m. would give,

$$\gamma^\mu \partial_\mu \psi = -im\psi$$
$$\partial_\mu \bar{\psi} \gamma^\mu = im\bar{\psi}$$

$$\Rightarrow \partial_\mu j^\mu = (\partial_\mu \bar{\psi})\gamma^\mu \psi + \bar{\psi}\gamma^\mu \partial_\mu \psi$$
$$= (im\bar{\psi})\psi + \bar{\psi}(-im\psi) = 0 \quad \checkmark$$

$$\partial_\mu j^{\mu 5} = (\partial_\mu \bar{\psi})\gamma^\mu \gamma^5 \psi + \bar{\psi}\gamma^\mu \gamma^5 \partial_\mu \psi$$
$$= (im\bar{\psi})\gamma^5 \psi - \bar{\psi}\gamma^5(-im\psi)$$
$$= 2im\bar{\psi}\gamma^5\psi \neq 0$$

where $\{\gamma^5, \gamma^\mu\} = 0$

Therefore, $\psi \to e^{i\alpha}\psi$ is a symmetry of $\mathcal{L}_D(m=0), \mathcal{L}_D(m\neq 0)$ whereas the so-called "chiral transformation", $\psi \to e^{i\alpha\gamma^5}\psi$ is only a symmetry for $\mathcal{L}_D(m=0)$.

Chiral Anomalies
Chiral Anomaly Cancellation, SM

An anomaly arrises when a classically conserved current or classical symmnetry, in classical field theory, is no longer held under quantum field theory due to quantum fluctuations.

In other words, what was once $\partial_\mu j^\mu = 0$ and $k_\mu M^{\mu \cdots}(k, \dots) = 0$ is sometimes no longer due to quantum fluctuations.

$\boxed{\text{e.g. } \partial_\mu j^\mu \neq 0}$

An important anomaly occurrence is the so-called chiral anomaly, which can be demonstrated through direct calculation of triangle diagrams.

Chiral Anomaly Example

– A. Zee, "QFT in a Nutshell" pgs. 270-278

Consider the following single massless fermion theory, $\mathcal{L} = \bar{\psi} i \gamma^\mu \partial_\mu \psi$

$$\left(\mathcal{L}_{Dirac} = \bar{\psi}(i\gamma^\mu \partial_\mu - \cancel{m})\psi \right)$$

No gauge field couplings, could add later if we wanted.

Note: $\mathcal{L}$ is invariant under

$$\psi \to e^{i\alpha}\psi \text{ and } \psi \to e^{i\alpha\gamma^5}\psi$$

$$\Rightarrow j^\mu = \bar{\psi}\gamma^\mu\psi \text{ and } j^{\mu 5} = \bar{\psi}\gamma^\mu\gamma^5\psi$$

(conserved vector current) (conserved axial current)

With E.L. e.o.m.,

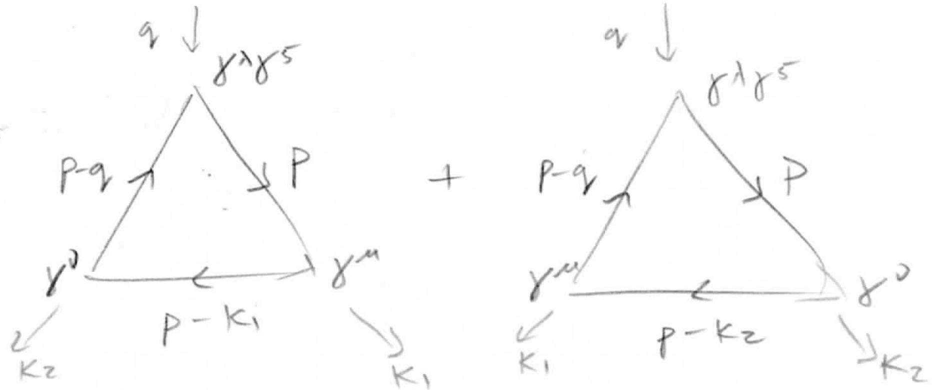
$$i\gamma^\mu \partial_\mu \psi = 0, \quad -i\partial_\mu \bar{\psi}\gamma^\mu = 0,$$

one can verify classically

$$\partial_\mu j^\mu = 0, \quad \partial_\mu j^{\mu 5} = 0$$

Now lets calculate an amplitude consisting of 2-triangle (loop) diagrams to see if our classical conservations still hold.

$iM^{\lambda\mu\nu}(k_1,k_2) =$



(two due to Bose statistics)

fermion loop, propagators

$$= (-1)\, i^3\, tr\left[\gamma^\lambda\gamma^5 \frac{1}{\not{p}-\not{q}}\gamma^\nu\frac{1}{\not{p}-\not{k}_1}\gamma^\mu\frac{1}{\not{p}} + \gamma^\lambda\gamma^5\frac{1}{\not{p}-\not{q}}\gamma^\mu\frac{1}{\not{p}-\not{k}_2}\gamma^\nu\frac{1}{\not{p}}\right]$$

where $q = k_1 + k_2$

Note: If $\partial_\mu j^\mu = 0$ and $\partial_\mu j^{\mu 5} = 0$ hold then $k_{1\mu} M^{\lambda\mu\nu} = k_{2\nu} M^{\lambda\mu\nu} = 0$ and $q_\lambda M^{\lambda\mu\nu} = 0$ respectively.

Failure to conserve our vector currents results in fermion charge not being conserved, i.e. $Q = \int d^3x\, J^0 \neq$ constant, and undo gauge invariance for any gauge field couplings to be considered; e.g. A_μ has only 2 degrees of polarization, need $k_{1\mu} M^{\lambda\mu\nu} = 0$ such that the gauge dependent term $\xi(k_{1\mu}k_{1\rho}/k_1^2)$ of the propagator

$$\left(\frac{i}{k_1^2}\right)\left[\xi\frac{k_{1\mu}k_{1\rho}}{k_1^2} - g_{\mu\rho}\right]$$ vanishes.

Moreover if the Ward-Identity fails, then the respective current is not conserved (e.g. if $k_{1\mu} M^{\lambda\mu\nu} \neq 0$, then $\partial_\mu j^\mu \neq 0$)

Let's check if $k_{1\mu} M^{\lambda\mu\nu} \stackrel{?}{=} 0$. (note: $\int \frac{d^4p}{(2\pi)^4} \equiv \int \not{d}^4 p$)

$$k_{1\mu} M^{\lambda\mu\nu} = i \int \not{d}^4 p \, \mathrm{tr}\left[\gamma^\lambda \gamma^5 \frac{1}{\not{p}-\not{q}} \left(\gamma^\nu \frac{1}{\not{p}-\not{k}_1} \not{k}_1 - \not{k}_1 \frac{1}{\not{p}-\not{k}_2} \gamma^\nu \right) \frac{1}{\not{p}} \right]$$

Using $\not{k}_1 \to \not{p} - (\not{p}-\not{k}_1)$ and
$\not{k}_1 \to (\not{p}-\not{k}_2) - (\not{p}-\not{q})$ in the 1st & 2nd terms respectively, we simplify to

$$k_{1\mu} M^{\lambda\mu\nu}(k_1,k_2)$$
$$= i \int \not{d}^4 p \, \mathrm{tr}\left[\gamma^\lambda \gamma^5 \frac{1}{\not{p}-\not{q}} \gamma^\nu \frac{1}{\not{p}-\not{k}_1} - \gamma^\lambda \gamma^5 \frac{1}{\not{p}-\not{k}_2} \gamma^\nu \frac{1}{\not{p}} \right]$$

Notice: If we shift $p \to p - k_1$ in the 2nd term, it appears that $k_{1\mu} M^{\lambda\mu\nu} = 0$ (not always the case, and not here).

When is $\int_{-\infty}^{\infty} dp\, f(p+a) = \int_{-\infty}^{\infty} dp\, f(p)$

or rather $\int_{-\infty}^{\infty} dp\, f(p+a) - f(p) = 0$?

To determin whether this can be done or not, consider the expansion of $f(p+a)$ in powers of a,

$$f(p+a) = \sum_{n=0}^{\infty} \frac{1}{n!} a^n \frac{d^n}{dp^n} f(p)$$

$$= f(p) + a \frac{df(p)}{dp} + \frac{a}{2} \frac{d^2 f(p)}{dp^2} + \cdots$$

$$\Rightarrow \int_{-\infty}^{\infty} dp \left(a \frac{d}{dp} f(p) + \cdots \right) = a\left(f(\infty) - f(-\infty) + \cdots\right)$$

if $f(\infty) = c_1$, $f(-\infty) = c_2$, but $c_1 \neq c_2$

not okay.

It depends on the rates of convergence and or divergence.

$\therefore$ Must check for significant terms using this expansion method.

Back to our case, we Wick rotate to Euclidean Space,

$$\int d_E^d p \left[f(p+a) - f(p) \right] = \int d_E^d p \left[a^\mu \partial_\mu f(p) + \cdots \right]$$

Using Gauss's Theorem $\left(\int_V \vec{\nabla} \cdot \vec{F}\, dV = \oint_S \vec{F} \cdot \hat{n}\, dS \right)$

we get,

$$\lim_{P \to \infty} a^\mu \left(\frac{P_\mu}{P} \right) f(P)\, S_{d-1}(P)$$

where $S_{d-1}(P)$ is the area of a $(d-1)$-dimensional sphere (infinitely large) and an average over the surface of the sphere is understood.

Wick rotating back (now in $d=4$)

$$\int d^4 p \{ f(p+a) - f(p) \} = \lim_{P \to \infty} i a^\mu \left(\frac{P_\mu}{P} \right) f(P) \left(2\pi^2 P^3 \right)$$

Noting,

$$f(p) = tr\left[\gamma^\lambda \gamma^5 \frac{1}{\not{p} - \not{k}_2} \gamma^\nu \frac{1}{\not{p}} \right] = tr[\gamma^5 \gamma^\tau \gamma^\nu \gamma^\sigma$$

$$= tr\left[\gamma^5 \gamma^\tau \gamma^\nu \gamma^\sigma \gamma^\lambda \right] \frac{(p-k_2)_\tau\, p_\sigma}{[(p-k_2)^2 - i\epsilon][p^2 - i\epsilon]} = \frac{4i\epsilon^{\tau\nu\sigma\lambda} k_{2\tau} p_\sigma}{[(p-k_2)^2 - i\epsilon][p^2 - i\epsilon]}$$

$$\Rightarrow k_{1\mu} M^{\lambda\mu\nu} = \frac{i}{(2\pi)^4} \lim_{P \to \infty} i(-k_1)^\mu \frac{P_\mu}{P} \frac{4i\epsilon^{\tau\nu\sigma\lambda} k_{2\tau} P_\sigma}{P^4} 2\pi^2 P^3 = \frac{i}{8\pi^2} \epsilon^{\lambda\nu\tau\sigma} k_{1\tau} k_{2\sigma}$$

$$\neq 0$$

Turns out, anomaly is happens to be spread out among each and all three currents, e.g. $k_{2\nu}M^{\lambda\mu\nu} \neq 0$, $q_{\lambda}M^{\lambda\mu\nu} \neq 0$

Can shift reference frame, $p \to p+a$ such that we at least ensure conservation of our vector currents (which could also couple to gauge fields; want $Q =$ constant, who cares about $Q_5 =$ constant.)

Then

$M^{\lambda\mu\nu}(a, k_1, k_2)$

$$= (-i)\, i^3 \int d^4p \; tr\left[\gamma^\lambda \gamma^5 \frac{1}{\not p + \not a - \not q} \gamma^\nu \frac{1}{\not p + \not a - \not k_1} \gamma^\mu \frac{1}{\not p + \not a} \right]$$

$$+ \; \{ \mu, k_1 \leftrightarrow \nu, k_2 \}$$

Computing,

$M^{\lambda\mu\nu}(a, k_1, k_2) - M^{\lambda\mu\nu}(k_1, k_2)$ with

$$f(p) = \lim_{p \to \infty} \frac{tr[\gamma^\lambda \gamma^5 \not p \gamma^\nu \not p \gamma^\mu \not p]}{p^6}$$

$$= \frac{2p^\mu tr[\gamma^\lambda \gamma^5 \not p \gamma^\nu \not p] - p^2 tr[\gamma^\lambda \gamma^5 \not p \gamma^\nu \gamma^\mu]}{p^6}$$

$$= + \frac{4i\, p^2 p_\sigma \epsilon^{\sigma\nu\mu\lambda}}{p^6}$$

$\Rightarrow \; M^{\lambda\mu\nu}(a, k_1, k_2) - M^{\lambda\mu\nu}(k_1, k_2)$

$$= \frac{4i}{8\pi^2} \lim_{p \to \infty} a^\omega \frac{p_\omega p_\sigma}{p^2} \epsilon^{\sigma\nu\mu\lambda} + \{ \mu, k_1 \leftrightarrow \nu, k_2 \}$$

$$= \frac{i}{8\pi^2} \epsilon^{\sigma\nu\mu\lambda} a_\sigma + \{ \mu, k_1 \leftrightarrow \nu, k_2 \}$$

Choosing $a = \alpha(k_1 + k_2) + \beta(k_1 - k_2)$

$$M^{\lambda\mu\nu}(a, k_1, k_2) = M^{\lambda\mu\nu}(k_1, k_2) + \frac{i\beta}{4\pi^2} \epsilon^{\lambda\mu\nu\sigma}(k_1 - k_2)_\sigma$$

where α drops out.

Demanding $k_{1\mu} M^{\lambda\mu\nu}(a, k_1, k_2) = 0$,

fixes β.

since $k_{1\mu} M^{\lambda\mu\nu}(k_1, k_2) = \frac{i}{8\pi^2} \epsilon^{\lambda\nu\tau\sigma} k_{1\tau} k_{2\sigma}$

$$\Rightarrow \frac{i}{8\pi^2} \epsilon^{\lambda\nu\tau\sigma} k_{1\tau} k_{2\sigma} = -\frac{i\beta}{4\pi^2} \epsilon^{\lambda\mu\nu\sigma} k_{1\mu} (k_1 - k_2)_\sigma$$

$$= \frac{-i\beta}{4\pi^2} (+\epsilon^{\lambda\nu\tau\sigma}) k_{1\tau} (+k_2)_\sigma$$

$\therefore \beta = -\frac{1}{2}$, $a = -\frac{1}{2}(k_1 - k_2)$

Lastly,

$$q_\lambda M^{\lambda\mu\nu}(k_1, k_2) = i \int d^4p \; tr\left[\gamma^5 \frac{1}{\not{p} - \not{q}} \gamma^\nu \frac{1}{\not{p} - \not{k}_1} \gamma^\mu - \gamma^5 \frac{1}{\not{p} - \not{k}_2} \gamma^\nu \frac{1}{\not{p}} \gamma^\mu \right] + \{\mu, k_1 \leftrightarrow \nu, k_2\}$$

$$= \frac{i}{4\pi^2} \epsilon^{\mu\nu\lambda\sigma} k_{1\lambda} k_{2\sigma}$$

$$\Rightarrow \boxed{q_\lambda M^{\lambda\mu\nu}(a, k_1, k_2) = \frac{i}{2\pi^2} \epsilon^{\mu\nu\lambda\sigma} k_{1\lambda} k_{2\sigma}}$$

$\partial_\lambda j^{\lambda 5} \neq 0$ axial current is not conserved

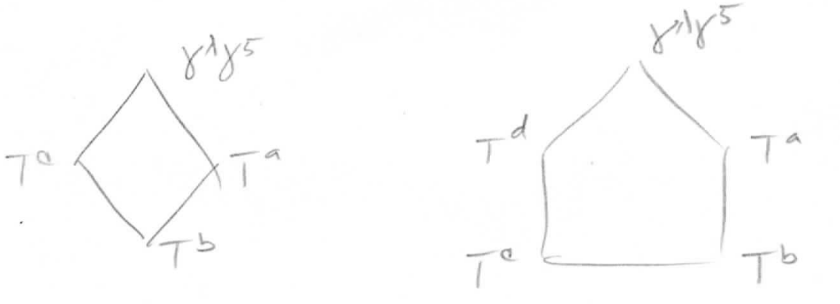
Note: Suppose $\mathcal{L} = \bar{\psi} i \gamma^\mu (\partial_\mu - i g A^a_\mu T^a)\psi$ (nonabelian theory)

The vertices would contain group factors.

Moreover, $\partial_\mu J_5^\mu$ generalizes to

$$\partial_\mu J_5^\mu = \frac{g^2}{(4\pi)^2} \underbrace{\epsilon^{\mu\nu\lambda\sigma} \, \mathrm{tr}\, F_{\mu\nu} F_{\lambda\sigma}}_{\text{contains terms quadratic, cubic, and quartic in } A.}$$

Thus, in addition to the triangle anomaly, the square and pentagon anomaly also exist.



Lastly,
One can imagin ~~this~~ theory containing other fermion couplins such as to a scalar field $f\phi\bar{\psi}\psi$.

3-loops

$$\partial_\mu J_5'^{\mu} = \partial_\mu J_5^\mu (1 + h(\underset{\uparrow}{f, e, \ldots}))$$

all couplings in new theory

Adler and Bardeen proved $h = 0$.

Additional propagators make shifts acceptable, W.I.'s are satisfied.

Chiral anomaly an important consideration of the S.M. due to $SU(3)_c \times \underbrace{SU(2)_L} \times U(1)_Y$

- Weak Interactions only interact with left handed fermions.

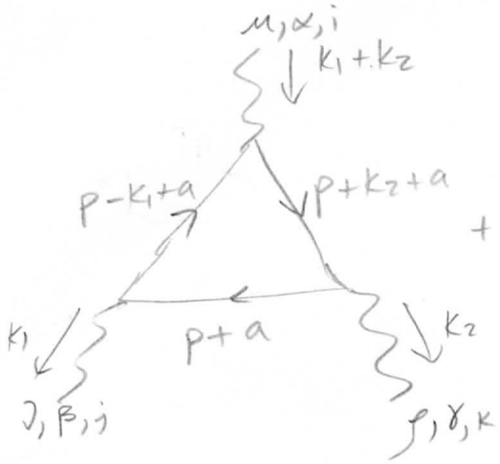
General Case: (Non-Abelian SU(N))

 $= \frac{i}{\not p}$;  $= -ig_i T_{i\alpha} \gamma^\mu P_R$

no sum on i

i represents a particular gauge group

Right-Hand First

$iM^{\mu\nu\rho}_{\alpha,\beta,\gamma,i,j,k} =$  + 

$$= (-1)(-ig_i)(-ig_j)(-ig_k)\, i^3 \int d^4p$$

$$* \, tr\left[T_{\beta j} T_{\gamma k} T_{\alpha i}\right] tr\left[\frac{1}{\not p - \not k_1 + \not a}\gamma^\nu \frac{1}{\not p + \not a}\gamma^\rho \frac{1}{\not p + \not k_2 + \not a}\gamma^\mu P_R\right]$$

$$+ \, tr\left[T_{\gamma k} T_{\beta j} T_{\alpha i}\right] tr\left[\frac{1}{\not p - \not k_2 + \not b}\gamma^\rho \frac{1}{\not p + \not b}\gamma^\nu \frac{1}{\not p + \not k_1 + \not b}\gamma^\mu P_R\right]$$

Will choose a, b such that only one current is chiral-anomalous.

$$k_{1\mu} M^{\mu\nu\rho}_{\alpha,\beta,\gamma,i,j,k} = ?$$

Simplify integrals by using

$$\not{k}_1 + \not{k}_2 = (\not{p} + \not{k}_2 + \not{a}) - (\not{p} - \not{k}_1 + \not{a})$$
$$= (\not{p} + \not{k}_1 + \not{b}) - (\not{p} - \not{k}_2 + \not{b})$$

$$k_{1\mu} M^{\mu\nu\rho}_{\alpha\beta\gamma ijk} = g_i g_j g_k \int d^4p$$

$$\times \Big\{ tr[T_\beta T_\gamma T_\alpha]\, tr\Big[\frac{1}{\not{p}-\not{k}_1+\not{a}}\gamma^\nu \frac{1}{\not{p}+\not{a}}\gamma^\rho P_R\Big]$$

$$- tr[T_\beta T_\gamma T_\alpha]\, tr\Big[\frac{1}{\not{p}+\not{a}}\gamma^\rho \frac{1}{\not{p}+\not{k}_2+\not{a}}\gamma^\nu P_R\Big]$$

$$+ tr[T_\gamma T_\beta T_\alpha]\, tr\Big[\frac{1}{\not{p}-\not{k}_2+\not{b}}\gamma^\rho \frac{1}{\not{p}+\not{b}}\gamma^\nu P_R\Big]$$

$$- tr[T_\gamma T_\beta T_\alpha]\, tr\Big[\frac{1}{\not{p}+\not{b}}\gamma^\nu \frac{1}{\not{p}+\not{k}_1+\not{b}}\gamma^\rho P_R\Big]\Big\}$$

Note: $tr[T_\beta T_\gamma T_\alpha] = D_{\alpha\beta\gamma} + \frac{1}{2} i N C_{\alpha\beta\gamma}$

$tr[T_\gamma T_\beta T_\alpha] = D_{\alpha\beta\gamma} - \frac{1}{2} i N C_{\alpha\beta\gamma}$

where $D_{\alpha\beta\gamma} = \frac{1}{2} tr[\{T_\alpha, T_\beta\} T_\gamma]$

and N is defined by $tr[T_\alpha T_\beta] = N\delta_{\alpha\beta}$

The anti-symmetric (group index) terms are non-zero but they don't break the symmetry (they're non-anamolous, see pg 374 of Vol. II of Weinberg's "The Quantum Theory of Fields")

Anomaly is contained in the symmetric part.

$$\Rightarrow \quad K_{1\mu} M^{\mu\nu\rho}_{\alpha\beta\gamma ijk} = \frac{g_i g_j g_k}{2} \, tr\left[\{T_{\alpha i}, T_{\beta j}\} T_{\gamma k}\right]$$

$$\times \{ \ \cdots \ \}$$

In the end (after shifts etc. pgs. 374-377 Weinberg II)

$$K_{1\mu} M^{\mu\nu\rho}_{\alpha\beta\gamma ijk} \propto A g_i g_j g_k \, tr\left[T_i^{\alpha}\{T_j^{\beta}, T_k^{\gamma}\}\right]$$

$$* \ \epsilon^{\nu\rho\tau\sigma} K_{1\tau} K_{2\sigma}$$

(Again one can add in the anomalous terms from the square and pentagon diagrams)

Anomaly cancellation in the SM:

Following along the same steps given towards the end of QFT B Spring 2013 (Dr. Okui), we find the following outcomes:

SM $SU(3)_c \times SU(2)_L \times U(1)_Y$

Must consider all possible (permutations of) couplings to triangle diagram vertices.

1.) First note for QED, $i=j=k=U(1)_{QED}$


$$= Ae^3 tr[Q(QQ+QQ)]\epsilon^{\nu\rho\tau\sigma} k_{1\tau} k_{2\sigma}$$

but  $= -Ae^3 tr[Q(QQ+QQ)]\epsilon^{\nu\rho\tau\sigma} k_{1\tau} k_{2\sigma}$

$\therefore$ anomalies cancel

2.) SM

	$SU(3)_c$	$SU(2)_L$	$U(1)_Y$
q_L	3	2	1/6
u_R	3	1	2/3
d_R	3	1	−1/3
e_L	—	2	−1/2
e_R	—	1	−1

– Consider $\{SU(3)_c\}^3$, $i = j = k = SU(3)_c$

$$q_\lambda M^{\lambda \ldots} \propto -2 g_3^3 \, \mathrm{tr}\big[\underbrace{T^a \{T^b, T^c\}}_{\text{Gell-mann matrices}}\big] \quad (\text{from } q_L) \;\text{doublet}$$

$$+ 2 g_3^3 \, \mathrm{tr}\big[T^a \{T^b, T^c\}\big] \quad (\text{from } u_R, d_R) \;\text{singlets}$$

$$= 0$$

– $\{SU(2)_L\}^3$ (only left-handed currents)

\# of colors (red, blue, green)

$$q_\lambda M^{\lambda \ldots} \propto -3 g_2^3 \, \mathrm{tr}\big[\tau^a \{\tau^b, \tau^c\}\big] \quad (\text{from } q_L)$$

$$- g_2^3 \, \mathrm{tr}\big[\tau^a \{\tau^b, \tau^c\}\big] \quad (\text{from } e_L)$$

$$\neq 0$$

however,

pauli spin matrices

$$\mathrm{tr}\big[\tau^a \{\tau^b, \tau^c\}\big] \propto \mathrm{tr}\big[\sigma^a \{\sigma^b, \sigma^c\}\big] = 2\delta^{bc} \, \mathrm{tr}[\sigma^a] = 0$$

– $\{U(1)_Y\}^3$

color comp. / $SU(2)_L$ components

$$q_\lambda M^{\lambda \ldots} \propto -3 \times 2 \times g_1^3 \left(\tfrac{1}{6}\right)^3$$

$$+ 3 * g_1^3 \left(\tfrac{2}{3}\right)^3$$

$$+ 3 * g_1^3 \left(-\tfrac{1}{3}\right)^3$$

$$- 2 * g_1^3 \left(-\tfrac{1}{2}\right)^3$$

2 $SU(2)_L$ comp.

$$+ 1 * g_1^3 (-1)^3$$

$$= 0$$

$$\left(\begin{array}{ll} q_L & -\frac{1}{6^2} \\ u_R & +\frac{8}{9} \\ d_R & -\frac{1}{9} \\ e_L & +\frac{1}{4} \\ e_R & -1 \end{array}\right) = 0$$

$- \{SU(3)_c\}^2 \times SU(2)_L$

$$q^\lambda M^{\lambda \ldots} \propto g_3{}^2 g_2 \, tr[\tau^a \{T^b, \tau\}]$$
$$= g_3{}^2 g_2 \, tr[\tau^a] \, tr[\{T^b, T^c\}]$$
$$= 0 \quad \text{since} \quad tr[\tau^a] = \tfrac{1}{2} tr[\sigma^a] = 0$$

pauli matrices are traceless.

$- \{SU(3)_c\}^2 \times U(1)_Y$

color; 2 SU(2) comp.

$$q^\lambda M^{\lambda \ldots} \propto -3 * 2 \, g_3{}^2 g_1 \, tr[\tfrac{1}{6}\{T^b, T^c\}] \quad \text{(for } q_L) \quad 3(-\tfrac{1}{3}$$
$$+ 3 * g_3{}^2 g_1 \, tr[\tfrac{2}{3}\{T^b, T^c\}] \quad \text{(for } u_R) \quad +\tfrac{2}{3}$$
$$+ 3 * g_3{}^2 g_1 \, tr[-\tfrac{1}{3}\{T^b, T^c\}] \quad \text{(for } d_R) \quad -\tfrac{1}{3})$$
$$= 0$$

$$= 0$$

$- \{SU(2)_L\}^2 \times U(1)_Y$

3 colors; 2 SU(2) comp.

$$q^\lambda M^{\lambda \ldots} \propto -3 * 2 * g_2{}^2 g_1 \, tr[\tfrac{1}{6}\{\tau^a, \tau^b\}] \quad \text{(for } q_L)$$
$$- 2 * g_2{}^2 g_1 \, tr[(-\tfrac{1}{2})\{\tau^a, \tau^b\}] \quad \text{(for } e_L)$$
$$= 0$$

$- SU(3)_c \times SU(2)_L \times U(1)_Y$

$$q^\lambda M^{\lambda \ldots} \propto \sum_{\substack{\text{color,} \\ SU(2)_L \text{ doublets,} \\ Y.}} g_3 g_2 g_1 \, tr[Y \{\tau^b, T^c\}] = 0$$

pauli & Gell-Mann matrices traceless.

zero bc $tr[\frac{\sigma^b}{2}] = tr[\frac{\lambda^c}{2}] = 0$