

THE RENORMALIZATION GROUP AND THE ϵ EXPANSION*

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Abstract:

The modern formulation of the renormalization group is explained for both critical phenomena in classical statistical mechanics and quantum field theory. The expansion in $\epsilon = 4 - d$ is explained [d is the dimension of space (statistical mechanics) or space-time (quantum field theory)]. The emphasis is on principles, not particular applications. Sections 1–8 provide a self-contained introduction at a fairly elementary level to the statistical mechanical theory. No background is required except for some prior experience with diagrams. In particular, a diagrammatic approximation to an exact renormalization group equation is presented in sections 4 and 5; sections 6–8 include the approximate renormalization group recursion formula and the Feynman graph method for calculating exponents. Sections 10–13 go deeper into renormalization group theory (section 9 presents a calculation of anomalous dimensions). The equivalence of quantum field theory and classical statistical mechanics near the critical point is established in section 10; sections 11–13 concern problems common to both subjects. Specific field theoretic references assume some background in quantum field theory. An exact renormalization group equation is presented in section 11; sections 12 and 13 concern fundamental topological questions.

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1. Introduction

The purpose of this paper is to discuss recent work on the renormalization group and its applications to critical phenomena and field theory. These ideas are illustrated using the other recent idea of defining critical phenomena and field theory in a space of dimension $4 - \epsilon$ (space-time dimension $4 - \epsilon$ for field theory) and expanding in powers of ϵ . The emphasis is on critical phenomena; basic ideas will be stressed rather than special results. The presentation is incomplete; this review is not a substitute for current literature. The first section is general and philosophical. Most of the subsequent sections are more pragmatic, being concerned with specific problems and calculations.

Associated with this section there is a list of recent references on the renormalization group and the ϵ expansion.

For a precise list of topics discussed in this paper, see the contents.

1.1. *The renormalization group and coherence problems in physics*

In this section a philosophical discussion of the renormalization group will be given. (One should reread this introduction after studying the rest of the paper.) Toward the end of the first section we begin the review of critical phenomena.

The renormalization group is a method for dealing with some of the most difficult problems of physics. These problems include relativistic quantum field theory, critical phenomena, the Kondo effect [e.g. 1–7] and others. These problems are all characterized by involving a large number of degrees of freedom, in an essential way.

Most of the problems one deals with in physics involve a very large number of degrees of freedom. For example, a crystal, liquid, or gas in macroscopic quantities involves more than 10^{23} electrons, and each coordinate of each electron is a degree of freedom.

In contrast, most theoretical methods work only when one has only one independent variable, i.e. only one degree of freedom. For example, consider the Schrödinger equation for a wave function $\psi(x, y, z)$ for one electron. It is infinitely easier to calculate ψ if one can separate variables in the Schrödinger equation (e.g., write $\psi = \psi_1(r)\psi_2(\theta)\psi_3(\phi)$ in spherical polar coordinates). It is obviously hopeless to compute a wave function for 10^{23} electrons without extraordinary simplifications, justified or otherwise.

Under normal circumstances the 10^{23} or so degrees of freedom can be reduced enormously. The intensive or extensive character of observables (energy is extensive, density is intensive) allows one to reconstruct the properties of a macroscopic system given only a microscopic sample of it. Thus a liquid of only 1000 atoms, say, would probably have approximately the same energy per unit volume and density as the same liquid (at the same temperature and pressure) with 10^{23} atoms.

How far can one reduce the size of a gas, say, without qualitatively changing its properties? The minimum size one can reach without change is called the correlation length. The correlation length ξ depends on the state of the system. For a gas ξ depends on pressure and temperature. In favorable circumstances ξ is only one or two atomic spacings. When ξ is this small there exist a variety of methods for calculating properties of the system: virial expansions, perturbation expansions, Hartree-Fock methods, etc. These methods involve a variety of approximations; but they have one feature in common. They all assume that the properties of matter in bulk can be related to the properties of small clusters of atoms. They involve further assumptions because even a cluster of only three atoms involves too many degrees of freedom to be soluble without considerable simplification.

In special cases the correlation length is much larger than the atomic spacing. The critical point marking the onset of a phase transition is a prime example. Liquid-gas transitions, ferromagnetic transitions, order-disorder transitions in alloys, etc., all exhibit critical points for special values of the thermodynamic variables. (The liquid-gas critical point occurs for critical values T_c and P_c of temperature and pressure.) Precisely at the critical point, ξ is infinite; near the critical point ξ is large.

There are a class of problems, including critical phenomena, which are characterized by having very many degrees of freedom in a region the size of a correlation length. "Very many" means not just 3 or 4, but hundreds or millions if not an infinite number. Other problems of this type besides critical phenomena are the Kondo problem (a magnetic impurity in a metal), the binding of large molecules, and the entire subject of relativistic quantum field theory. In the case of a quantum field, say $\phi(x)$, the field ϕ at each point x is a separate degree of freedom, so any region of finite size contains an infinite number of degrees of freedom. The correlation length of a quantum field is usually the Compton wavelength of the lowest mass particle. In the case of quantum electrodynamics it is the electron Compton wavelength (10^{-11} cm) rather than the photon Compton wavelength (∞) which acts as the correlation length in practice. One can relatively easily relate quantum electrodynamics in a box of size $> 10^{-11}$ cm to quantum electrodynamics in all space. Boxes of size $\ll 10^{-11}$ cm cause gross distortions of the interactions of electrons and photons.

The problems listed above are all noted for their intransigence. The binding of molecules is hardly better understood today than it was in 1932. There has been fitful progress in critical phenomena over the last 70 years. There has been sensational progress in calculating quantum electrodynamics, but very little progress in understanding it; and strong interactions are neither calculable nor understood. The Kondo problem has only recently been studied, and may be closer to solution [2, 3, 7].

Studies of renormalization in quantum field theory and critical phenomena in statistical mechanics both suggest that the behavior of systems with many degrees of freedom within a correlation length is qualitatively different from those with only a few degrees of freedom in a correlation length. The systems we are interested in are usually defined by means of a Hamiltonian, and one would normally have expected that the behavior of the system is determined mainly by the type of interactions present in the Hamiltonian and the strengths of the corresponding coupling constants. This is certainly the case when ξ is small. However, in the problems discussed here, where many degrees of freedom are behaving cooperatively, it appears that the behavior of the system is determined primarily by the fact that there is cooperative behavior, plus the nature of the degrees of freedom themselves. The interaction Hamiltonian plays only a secondary role. Thus, in critical phenomena there has developed the notion of universality, i.e. that all interaction Hamiltonians show the same critical behavior. The idea of universality originates in the "law of corresponding states", which is the hypothesis that all fluids and gases have the same equation of state apart from a renormalization of length and energy scales. For a comparatively recent reference on this law see Guggenheim [8]. Recently the idea of universality has been formulated more generally to relate critical behavior in different systems with arbitrary interactions. See for example Kadanoff [9]. Universality will be discussed further below and in section 12.

first objective:
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length.

The renormalization group approach has two objectives. The first is the practical one of simplifying the task of solving systems with many degrees of freedom contained within a correlation length. The basic idea is the same as in hydrodynamics. In hydrodynamics one introduces new variables such as the density $\rho(x)$ which represents an average over the original microscopic degrees of freedom. All microscopic fluctuations are eliminated in the hydrodynamic equations;

$\rho(x)$ is supposed to show only macroscopic fluctuations. What this means in effect is that the hydrodynamic degrees of freedom are the values of $\rho(x)$ at macroscopically separated points. Thus, there are far fewer hydrodynamic degrees of freedom per unit volume than original microscopic degrees of freedom per unit volume.

The renormalization group approach is similar to the hydrodynamic approach in that again the original microscopic degrees of freedom are replaced by a smaller set of effective degrees of freedom. This is done in steps; in each step the linear density of degrees of freedom is reduced by a factor 2.

Reducing the "density of degrees of freedom" can be realized in several ways. Suppose the original system was a system of spins with spacing L_0 . The new effective degrees of freedom might be spins with spacing $2L_0$. Alternatively the new degrees of freedom might be a magnetization density $M(x)$ depending on a continuous variable x (just as hydrodynamics replaces discrete atoms by a density $\rho(x)$). Then to limit the degrees of freedom one imposes the restriction that $M(x)$ only contains fluctuations with wavelengths greater than $2L_0$. These ideas will be formulated more precisely in later sections of this review. (The factor of 2 is arbitrary. In particular, it is often useful to change the density of degrees of freedom only infinitesimally instead of by a factor 2. See later sections.)

This reduction in the degrees of freedom is carried out repeatedly. To start with, the spacing of degrees of freedom is L_0 ; after 1 step the spacing is $2L_0$, after 2 steps the spacing is $4L_0$, etc. One proceeds with further steps until the separation of effective degrees of freedom is of order the correlation length ξ . In each step one has to construct an effective interaction for the effective degrees of freedom, just as in hydrodynamics one has to construct the hydrodynamic equations for $\rho(x)$. The simplification of the renormalization group lies in the hope that these effective interactions $\mathcal{H}_1, \mathcal{H}_2$, etc. are *local* interactions, i.e. the interactions should couple directly only nearby degrees of freedom. This of course assumes that the initial interaction was local, but this is true for the problems of interest here. Thus we assume that the range of interaction in the initial interaction is of order L_0 even when $\xi \gg L_0$. The hope is that the range of interaction in \mathcal{H}_1 is of order $2L_0$, the range of interaction in \mathcal{H}_2 is of order $4L_0$, etc. The alternative is that the range of interaction is ξ even for \mathcal{H}_1 , which would be disastrous for the renormalization group approach. This disaster is avoided in the examples that have been worked out so far, but may be a problem in other cases.

If it is true that the range of interaction in \mathcal{H}_1 is only of order $2L_0$, then one can imagine (again by analogy to hydrodynamics) that the coupling constants in \mathcal{H}_1 can be determined by studying the behaviour of the system confined to a region of size (of order) $2L_0$; i.e. one does not have to discuss regions of size ξ . Thus one no longer has a huge number of degrees of freedom to worry about. To determine the coupling constants in \mathcal{H}_2 requires a larger region of size $4L_0$. However, the idea here is to determine \mathcal{H}_2 starting from \mathcal{H}_1 whose degrees of freedom are spaced by $2L_0$, instead of the original interaction \mathcal{H}_0 with spacing L_0 . Then one still has a limited number of degrees of freedom to consider. In the same way one constructs \mathcal{H}_3 from \mathcal{H}_2 , \mathcal{H}_4 from \mathcal{H}_3 , etc., until one obtains the \mathcal{H}_n for which $2^n L_0 \sim \xi$. At this stage one has only a few degrees of freedom per correlation length and the problem can hopefully be solved by other methods. In general, when calculating \mathcal{H}_l one has to consider a region of size $2^l L_0$, but one starts from \mathcal{H}_{l-1} whose degrees of freedom have spacing $2^{l-1} L_0$. Thus for any l one can consider a region containing only a few effective degrees of freedom.

Unfortunately, the general renormalization group techniques do not reduce the problem to 1 degree of freedom. One can easily imagine that one needs to consider 60 or more effective degrees of freedom when determining \mathcal{H}_l from \mathcal{H}_{l-1} . Suppose for example that the effective degrees of freedom in \mathcal{H}_{l-1} are discrete spins, and that one has to discuss a three-dimensional

(*) otherwise you couldn't use the R.G. technique, because you couldn't study the scaling properties of the system.

cube of width 4 spins: this cube will contain 64 spins. This is not very practical. One must simplify the calculation further so that \mathcal{H}_l can be computed from \mathcal{H}_{l-1} using only *one* degree of freedom. The practical applications of the renormalization group described in later lectures involve either special circumstances (the case $d = 4 - \epsilon$ with small ϵ) or crude approximations (the "approximate recursion formula") such that only one degree of freedom is needed for the calculation of \mathcal{H}_l . (See, however, the supplemental list of references.)

The second aim of the renormalization group approach is to explain how the qualitative features of cooperative behavior arise. In the renormalization group framework, these qualitative features result from the iterative character of the renormalization group. Namely, there is a transformation τ which converts \mathcal{H}_0 to \mathcal{H}_1 , \mathcal{H}_1 to \mathcal{H}_2 , etc. The transformation is the same whether one is constructing \mathcal{H}_1 from \mathcal{H}_0 or \mathcal{H}_2 from \mathcal{H}_1 ; in each case one is thinning the degrees of freedom by a factor 2. The only difference is in the lengthscale (L_0 versus $2L_0$) which is easily transformed away. So one has a transformation τ which is to be applied repeatedly:

$$\tau(\mathcal{H}_0) = \mathcal{H}_1, \quad \tau(\mathcal{H}_1) = \mathcal{H}_2, \quad \tau(\mathcal{H}_2) = \mathcal{H}_3 \quad \text{etc.} \quad (1.1)$$

This transformation is to be iterated n times where $2^n L_0$ is of order ξ . When ξ is large, the number of iterations is large.

When one has a transformation τ which is iterated many times, the simplest result we can obtain is that the sequence \mathcal{H}_l approaches a fixed point of τ , namely an interaction \mathcal{H}^* satisfying

$$\tau(\mathcal{H}^*) = \mathcal{H}^*. \quad (1.2)$$

This is what will happen in the examples discussed later in this review.

A fixed point of a transformation is a property of the transformation τ itself. That is, to find possible fixed point Hamiltonians \mathcal{H}^* one must solve the fixed point equation (1.2). These equations make no reference to the choice of initial Hamiltonian \mathcal{H}_0 .

The possible types of cooperative behavior, in the renormalization group picture, are determined by the possible fixed points \mathcal{H}^* of τ . Suppose for example that there are three fixed points \mathcal{H}_A^* , \mathcal{H}_B^* , and \mathcal{H}_C^* . Then one would have three possible forms of cooperative behavior. If a particular system has an initial interaction \mathcal{H}_0 , one has to construct the sequence $\mathcal{H}_1, \mathcal{H}_2$, etc. in order to find out which of \mathcal{H}_A^* , \mathcal{H}_B^* , or \mathcal{H}_C^* gives the limit of the sequence. If \mathcal{H}_A^* is the limit of the sequence, then the cooperative behavior resulting from \mathcal{H}_0 will be the cooperative behavior determined by \mathcal{H}_A^* . In this example the set of all possible initial interactions \mathcal{H}_0 would divide into three subsets (called "domains"), one for each fixed point. Universality would now hold separately for each domain. See section 12 for further discussion.

This is how one derives a form of universality in the renormalization group picture. It is not so bold as previous formulations [9]. Experience with soluble examples of the renormalization group transformation for critical phenomena shows that it generally has a number of fixed points, so one has to define domains of initial Hamiltonians associated with each fixed point, and only within a given domain is the critical behavior independent of the initial interaction.

There is no *a priori* requirement that the sequence \mathcal{H}_l approach a fixed point for $l \rightarrow \infty$. In principle the sequence for large l could show limit cycle, ergodic or turbulent behavior; in such cases it would be difficult to do much calculation. See [10] for an illustration of ergodic and turbulent behavior. But even if the sequence \mathcal{H}_l does not approach a fixed point, it is unlikely that \mathcal{H}_n for large n is a smooth function of the parameters in \mathcal{H}_0 . The trouble is that small changes in the parameters in \mathcal{H}_0 tend to be amplified or deamplified by the transformation τ , and when τ is iterated many times these amplification or deamplification factors become very large (one would guess of order \sqrt{n} from random walk arguments). Thus if u_0 is a coupling constant in \mathcal{H}_0 one would expect large ranges of u_0 which are deamplified (\mathcal{H}_n depends very little on u_0)

2. "Local" = netto, ben definito

separated by small ranges of u_0 which are amplified (\mathcal{H}_n changes very rapidly with u_0). In the case that all sequences approach fixed points, the ranges of u_0 for which amplification occurs are those values of u_0 for which \mathcal{H}_0 is near the boundary between two domains, while deamplification occurs when \mathcal{H}_0 is well inside a particular domain.

In summary, the basic ideas of the renormalization group are, first, to generate a sequence of effective local interactions \mathcal{H}_l . That is, if the spacing of degrees of freedom in \mathcal{H}_l is a_l then the interactions should have range a_l not ξ ; and by choosing $a_{l+1} = 2a_l$ one should be able to construct \mathcal{H}_{l+1} from \mathcal{H}_l considering regions of size about a_{l+1} rather than ξ . Secondly, the existence of a transformation τ which is iterated repeatedly to construct \mathcal{H}_l from \mathcal{H}_0 suggests that the nature of \mathcal{H}_l for large l will be largely or wholly determined by τ itself rather than \mathcal{H}_0 , thereby leading to at least a limited form of universality.

To conclude this introduction we outline briefly the history of the renormalization group method. This discussion is not complete and probably not very accurate. Landau [11] proposed a hydrodynamic approach to critical phenomena in the late 1930's. His specific theory gave the same results as earlier mean field theory, which is experimentally false. However, the renormalization group approach is best seen as a more sophisticated realization of Landau's ideas.

The specific ideas of the renormalization group approach appeared in two papers of the early 1950's, in connection with quantum field theory. A formulation of the renormalization group transformation τ was given by Stueckelberg and Petermann [12]. Gell-Mann and Low [13], in a remarkable paper, discussed the idea of a fixed point of the transformation and some of its implications: they showed that a unique value of the bare charge e_0 in quantum electrodynamics would correspond to all (sufficiently small) values of the renormalized charge e . This is the amplification effect discussed earlier. In the limit of infinite cutoff (i.e. in the limit of infinitesimal spacing of the original degrees of freedom) τ is iterated an infinite number of times. Thus one can get infinite amplification, e.g. a continuous range of values of e corresponding to one value of e_0 . The recipes for renormalization group calculations were reviewed in Bogoliubov and Shirkov [14].

The early work on the renormalization group had two defects. It had no calculable experimental consequences, so no one had to take it seriously. Secondly, the intuitive ideas were encased in a thick shell of formalism; it has required many years to peel off the shell.

In the 1960's an extraordinary paper by Kadanoff [15] on critical phenomena contained an intuitive discussion of the idea of thinning the degrees of freedom. Kadanoff assumed that one could discuss blocks of spins in a ferromagnet as if they were single effective spins with very simple interactions. Kadanoff showed that this assumption implied a set of "scaling laws" relating critical exponents which had been postulated earlier by Widom and others (see [16] for references). Kadanoff did not have a shred of justification for his assumption; the importance of his work was that it provided a simple but ideal picture of what an effective degree of freedom would be and how it would interact. This picture is surely unrealizable in practice. What one does is try to come as close as one can to this picture. One tries to define effective degrees of freedom which are roughly describable as block spins, and interactions which have a simple form (not necessarily Ising-like), at least approximately. In summary, Kadanoff has defined a much more profitable goal to work towards than the elaborate formalisms of the earlier work.

The differential equations of the renormalization group resurfaced in the Kondo problem in the work of Anderson et al. and others [2-7]. Anderson [4] was the first to derive these equations explicitly using the idea of reducing the number of degrees of freedom. The previous field theoretic work had a much less transparent justification.

Until recently all formulations of the renormalization group involved a transformation τ acting on a very restricted space of interactions. Gell-Mann and Low considered the standard electrodynamic interaction with the charge e_0 as the only free parameter. Kadanoff allowed only

nearest neighbor Ising-type interactions. A heavy price is paid for this restriction: one can only define τ when the solution of the problem is known. For example, Gell-Mann and Low define τ in terms of the exact electron propagator, which is not known until electrodynamics is solved. Thus there is only a limited possibility of using the renormalization group to help in solving the theory. A study of the fixed source model of the nucleon, in simplified form, showed that this problem disappears if one is willing to let \mathcal{H}_l contain all possible interactions, not just one or two [17]. It is then relatively easy to construct examples of the transformation τ without solving the theory (see later lectures). Now one pays a different price: one cannot keep track of all possible interactions at once because there are too many of them. So one has to have only a few dominant interactions if the renormalization group approach is to be practical. There can be many more interactions with small coupling constants, as long as these can be handled by perturbation methods. This was precisely the situation in the fixed source model. It is also true in the examples discussed later in this review, but it is not at all certain to be true in general.

Current work on the renormalization group will be summarized below.

There have been other ideas for dealing with problems such as critical phenomena. We mention specifically the Migdal-Polyakov bootstrap approach to critical phenomena and field theory [18–26] and the Johnson-Baker-Willey formulation of electrodynamics [27–29] because these ideas are sometimes confused with the renormalization group approach. There is no transformation like τ in either the Migdal-Polyakov or Johnson-Baker-Willey theories; these authors make no attempt to thin the degrees of freedom. So their ideas cannot be classified as renormalization group methods. In place of consciously reducing the number of degrees of freedom, these authors substitute a prayer that an infinite sum of graphs can be replaced by a calculable subset. See below for further comments.

Another recent development is the Callan-Symanzik equations [30, 31]. Closely related to the original Gell-Mann–Low formulation of the renormalization group, these equations are proving to be valuable tools for analyzing the short distance behavior of Feynman graphs [e.g. 32]. At present the Callan-Symanzik equations are too formal to be practical outside of perturbation theory. They will remain so unless some intuition can be added the way Kadanoff added an intuitive picture to the renormalization group approach.

1.2. Current references

Current literature on the renormalization group will now be classified, briefly. See also the supplemental list at the end of this review. First, papers on critical phenomena will be listed. The approximate renormalization group recursion formula is derived in [33]. Numerical solutions of the recursion formula for the Ising case are in [33]. Grover, Kadanoff, and Wegner solved the Heisenberg case numerically [34]. Grover solved the X – Y model numerically [35]. Baker [36] and Dyson [37] have described “hierarchical” models for which the recursion formula is exact. Golner [38] has proposed a form of the recursion formula which gives a non-zero value of η (η is defined in section 2). Golner [39] has solved the recursion formula for an example with three-fold symmetry. The ϵ expansion about 4 dimensions was developed by Fisher and Wilson [40]. A calculation of the scaling properties of all perturbations about the critical point, within the ϵ expansion, is given by Wegner [41]; Fisher and Pfeuty [42] investigate slightly anisotropic Heisenberg models. The recursion formula has been applied to tricritical points (such as ^3He – ^4He mixtures) by Riedel and Wegner [43]. The methods of ref. [33] have been applied to the problem of phase separation by Langer and Bar-on [44].

The Feynman graph method for calculating critical exponents in powers of ϵ (section 8) was developed in [45]. It was applied to the “excluded volume” problem by De Gennes [46].

Nickel [47] has calculated the exponent γ to order ϵ^3 . The ϵ expansion for the equation of state near the critical point has been obtained by Brézin, Wallace and Wilson [48], [49]. The charged and neutral Bose gas is discussed by Ma [50]. Exponents in the presence of long range forces are computed by Suzuki [51, 52] and Fisher, Ma, and Nickel [53].

The Gell-Mann–Low formulation of the renormalization group was used by Larkin and Khmel'nitskii [54] to discuss the logarithmic behavior of critical phenomena in four dimensions ($\epsilon = 0$) and uniaxial ferroelectrics in three dimensions. De Pasquale, Di Castro and Jona-Lasinio [55–57] and Migdal [58] have discussed qualitative implications for critical phenomena of the Gell-Mann–Low formulation. Di Castro [59] has used the Gell-Mann–Low theory to confirm some results of [45]. Wegner [60] has given an extensive discussion of all kinds of perturbations about the critical point with emphasis on corrections to the scaling laws, using the modern formulation (see also section 12). Several questions have been discussed by Hubbard [61]; the problem of liquids is discussed by Hubbard and Schofield [62]. The Gaussian model with long range forces is discussed by Niemeijer and Van Leeuwen [63].

The work of Dyson [37], Larkin and Khmel'nitskii [54] and Di Castro and Jona-Lasinio [55–57] preceded the work described in this report. For a good survey of critical phenomena just prior to the new developments, see the entire volume of ref. [56]. The new work is summarized briefly in [64].

Many of the references listed above report specific applications of the renormalization group approach which are omitted from this review.

There is another (earlier, but incomplete) paper (set of lecture notes) on the renormalization group [65]. They concern the ideas of the modern renormalization group approach with less emphasis on calculation than the present report.

The close mathematical analogy between critical phenomena and quantum field theory has been emphasized by Gribov and Migdal [66, 67] and Polyakov [68–70] in terms of Feynman diagrams; Moore [71] compares the Feynman path integral to the partition function; Suri [72] makes the connection using the transfer matrix formalism of statistical mechanics (see section 10).

For a review of work on the renormalization group in quantum field theory using the Gell-Mann–Low formulation see [73]. For a field theoretic formulation of the approximate recursion formula, see [74]. See likewise Golner [75].

The idea of a noninteger dimension d and an expansion about $d = 4$ is a useful theoretical device without physical significance: the physics is only in integer dimension d . Nonintegral dimensions will be introduced by analytic continuation in renormalization group equations (section 4) or Feynman graphs (section 8). The idea of nonintegral dimensions in statistical mechanics is not new: see [76] for example. More recently, Widom has studied $1 + \epsilon$ dimensions [77]. Noninteger d has been used recently in field theory to regularize Feynman graphs [78–80]; a thorough discussion is given in 't Hooft and Veltman [78]. Quantum field theory for noninteger d is discussed in [81] where the ϵ expansion is used to compute anomalous dimensions (see also section 9).

Another expansion technique in both critical phenomena and field theory is the $1/N$ expansion where N is the number of internal components of a spin (critical phenomena) or the number of internal components of a quantum field. Stanley [82] discovered that the limit $N \rightarrow \infty$ for a spin system reduces to the soluble spherical model of Berlin and Kac [83]. This was discussed further by Kac and Thomson [84]. Techniques for expanding exponents in $1/N$ were discovered by Abe [85] and Wilson [81]. They have been exploited by Abe and Hikami [86], [87], Suzuki [51], [52], Ma [50], [88], Fisher et al. [53], and Ferrell and Scalapino [89]. Brézin and Wallace [90] and Suzuki [52] have computed the equation of state using this expansion. The expansion is

probably not very useful for $N = 1$ to 3, the cases of most physical interest. This is because the ϵ expansion (valid for any N) has denominators of the form $N + 8$ (see section 8) suggesting the $1/N$ expansion is valid only for $N > 8$. However it is instructive to study models with large N as models in their own right. The application of the $1/N$ expansion to field theories in less than four space-time dimension is discussed in [81].

There is a special situation where the problem of a large number of degrees of freedom within a correlation length is trivial, namely free field theories (quantum field theory) or the Gaussian model in statistical mechanics (see section 3). These theories are trivial because the interaction can be diagonalized in momentum space. There exist a variety of graphical methods for treating small perturbations about these trivial theories. The ϵ expansion makes it possible to do successful calculations in critical phenomena by perturbation methods. There are now a bewildering variety of methods for setting up the ϵ expansion: besides the methods discussed in this paper, De Castro [59] has used the old Gell-Mann-Low methods and Mack [26] has used the Migdal-Polyakov approach. The new renormalization group method is emphasized in this paper because of its potential for handling nonperturbative problems (see refs. [33, 60] and sections 12 and 13). By comparison, for problems which cannot be reduced to a few diagrams, it is hopeless to calculate anything using the Gell-Mann-Low theory. Even for questions of principle the Gell-Mann-Low theory is not satisfactory because it involves assumptions which are technical and not physically motivated [73]. It is hard to believe these assumptions are always valid, although they appear to be true order by order in a diagrammatic expansion [e.g. 73]. The Migdal-Polyakov bootstrap (with conformal invariance assumed) is more interesting although not very practical unless only a few diagrams are important. For field theorists an interesting and unique feature of the Migdal-Polyakov bootstrap is Polyakov's "stream unitarity" [91].

The next seven sections are concerned with critical phenomena. This is partly because this is the most successful application of the renormalization group approach. However, critical phenomena provides the simplest and clearest example of the problem of many degrees of freedom within a correlation length. The experiments are precise and relatively unambiguous and test the fundamental aspects of the theory. The simplest models like the Ising model have no complications other than the basic problem of the large correlation length. The study of critical phenomena gives one an overview of the problem of many degrees of freedom within a correlation length that cannot be obtained any other way and is essential to understand the field theoretical applications discussed in sections 9-13.

There are excellent reviews of experiment and previous theory of critical phenomena (see section 2); this report will be devoted to explaining the renormalization group approach. Enough background will be supplied so one can read this review without previous background in critical phenomena, but the relation of the renormalization group to experiment and prior theory will not be discussed in detail.

Sections 11, 12, and 13 are of importance to statistical mechanics; but the main interest from section 9 on is quantum field theory. They depend heavily on the previous sections. They are part of a series of papers [92, 73] explaining the renormalization group approach as an alternative to canonical field theory. The ultimate aim is to produce a field theory of strong interactions, but breakthroughs are still required both in fundamental theory and techniques of calculation before the aim can be realized. Meanwhile, the study of the renormalization group approach has provided, as a byproduct, useful ideas which have been incorporated into more phenomenological approaches to strong interactions [92, 93]. Particle physicists who have studied the ideas of this paper find that it takes considerable effort over a period of time to understand them; they provide a stimulating point of view on the problems of quantum field theory; and the outlook for the future of these ideas is at present uncertain. See section 14.

1.3. Elementary facts about the Ising model

Let us begin by discussing the elementary features of the Ising model. (For the history of the Ising model, see [94].) The model has served in the past as a description of ferromagnetism but probably also constitutes a model of relativistic field theory. We imagine a cubic lattice of points, $n = (n_1, n_2, n_3)$ in three dimensions, say, and attach to each lattice point a spin variable s_n . We suppose that s_n takes on only the discrete values ± 1 , so that it is not treated as a full-fledged 3-dimensional quantum mechanical spin variable. If the lattice consists of N sites, the system clearly possesses 2^N possible states. Each of these spin configurations will have a particular energy. If only nearest neighbor spin-spin interactions are allowed the Hamiltonian is,

$$H = -J \sum_n \sum_i s_n s_{n+i} \quad \text{where } \{i\} \text{ are the unit vectors for each axis on the lattice (fig. 1.1). Furthermore, if the lattice is} \quad (1.3)$$

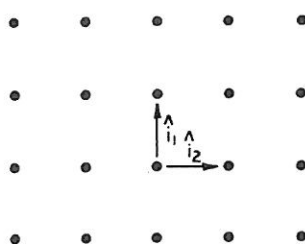


Fig. 1.1. A two-dimensional lattice, with unit vectors for each axis.

emersed in an external magnetic field B , the Hamiltonian acquires an additional term,

$$H = -J \sum_n \sum_i s_n s_{n+i} + \mu B \sum_n s_n \quad (1.4)$$

where μ is a certain gyromagnetic ratio. The thermodynamics of the system can be obtained from the partition function

$$Z = \sum_{\{\text{configurations}\}} e^{-H/kT} \quad (1.5)$$

where the sum runs over all possible spin configurations of the lattice, T is the temperature, and k is Boltzmann's constant. It is also convenient to introduce a free energy F ,

$$[F = -kT \ln Z.] \quad (1.6)$$

Consider also the magnetization per lattice site,

$$M = \frac{1}{N} \sum_n \frac{\langle \mu s_n \exp(-H/kT) \rangle}{\langle \exp(-H/kT) \rangle} \quad (1.7)$$

where $\langle \dots \rangle$ stands for a sum over configurations. M can be written

$$M = -\frac{1}{N} \frac{\partial}{\partial B} \left(\frac{F}{kT} \right) \quad \text{or} \quad M = \frac{1}{N} kT \frac{\partial}{\partial B} F \quad (1.8)$$

2. "full-fledged" = che ha ueno tutte le brone = che e in grado di volere (lett.);
che si e fatto le ossa (fig.).

The Ising model, in 2 or more dimensions, can display spontaneous magnetization. Let the lattice be in an external magnetic field B . If M remains different from zero when the external field is removed, there is spontaneous magnetization. For example, this occurs if at zero temperature the state of lowest energy of the system has all the spins aligned. We can expect $M = 1$ at zero temperature but $M = 0$ at infinite temperature, where thermal energy overwhelms the spin-spin interaction. Only for temperatures T less than a critical value T_c does spontaneous magnetization occur. Spontaneous magnetization is also closely related to the behavior of the spin-spin correlation function defined by,

$$\Gamma_n = \frac{\langle s_n s_0 \exp(-H/kT) \rangle}{\langle \exp(-H/kT) \rangle}. \quad (1.9)$$

At zero temperature when all the spins line up Γ_n is unity, independent of n . However, as $T \rightarrow \infty$ thermal excitations wash out the spin-spin correlations, so Γ_n is different from zero only at the origin. We will in fact see that the way Γ_n falls to zero as n increases will depend on whether $T < T_c$, $T = T_c$ or $T > T_c$. The Ising model, therefore, contains two lengths: the lattice spacing, and the distance over which spin-spin correlations are appreciable.

2 lengths < lattice spacing
distance over which spin-spin correlations are appreciable

In the first section we discussed the Ising model and the phenomenology of a critical point, such as the critical temperature at which spontaneous magnetization first occurs. We will study through seven sections the problem of critical behavior looking specifically at phenomena above the critical temperature where, although there is no spontaneous magnetization, the system feels the presence of the nearby critical point. In this section we will consider various theories of the critical point at the handwaving level: Landau's form of mean field theory, and the Kadanoff theory of effective interactions involving block spins. This second approach will provide the background for the renormalization group. (Some standard references on critical phenomena are refs. [95–101].)

2.1. Elementary properties of systems near their critical temperature

If we plot the spontaneous magnetization versus temperature for a ferromagnet, an anti-ferromagnet or the Ising model, one finds a curve similar to that shown in fig. 2.1. There is a critical temperature, T_c , at which spontaneous magnetization first occurs. The curve rises to some finite value at temperature zero. In the region just below the critical temperature, the magnetization is well approximated by a power law,

$$M \propto (T_c - T)^\beta \quad (2.1)$$

where β is an example of a critical exponent. Theories of critical behavior are mainly concerned with predicting such critical exponents or, at least, finding relations between them.

There is also the correlation length which we introduced earlier. Consider two spins, one at lattice site n , one at the origin and define the correlation function,

$$\Gamma_n = \langle s_n s_0 \exp(-H/kT) \rangle / \langle \exp(-H/kT) \rangle \quad (2.2)$$

where $\langle \dots \rangle$ indicates a sum over all configurations. The correlation function has the property

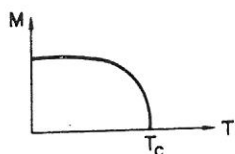


Fig. 2.1. A graph of magnetization versus temperature for a ferromagnet. T_c is the critical temperature.

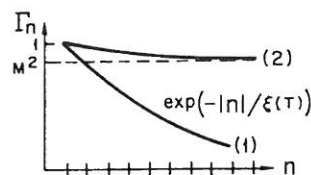


Fig. 2.2. Spin-spin correlation function versus distance between sites. Curve (1) is an example with $T > T_c$; curve (2) is an example with $T < T_c$.

that if $T > T_c$, then Γ_n falls off with $|n|$ (curve 1 in fig. 2.2). The asymptotic behavior for large $|n|$ is thought to be

$$\Gamma_n \sim \exp\{-|n|/\xi(T)\} \quad (2.3)$$

where $\xi(T)$ is the correlation length. (There are various ways of defining the correlation length. It is not evident that the qualitative definition of section 1 (see, in particular, the discussion p. 78) agrees with the precise definition of (2.3). They will in fact not always agree, but for the Ising model above T_c they do agree so far as is known.) Unfortunately, due to the complexity of the system, the validity of (2.3) has not been proved in general. For $T < T_c$, the correlation function is expected to approach the square of the magnetization divided by the square of the magnetic moment as n increases (curve 2, fig. 2.2). In other words, for large n actual correlation has disappeared and what remains is the product of the expectation values of the single spins. At the critical temperature the correlation function falls to zero for large n because there is no spontaneous magnetization. However, it falls very slowly (fig. 2.3) because we are at a transition point. The correlation function at T_c is expected to fall as a power of n and, by convention, is written in the form

$$\Gamma_n \sim 1/|n|^{d-2+\eta} \quad (2.4)$$

where d is the dimensionality of the system. In two dimensions we know that (2.4) is the correct form of the correlation function [95]. However, in three dimensions the statement is just a guess. η defines a second critical exponent.

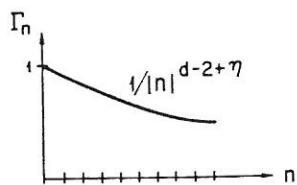


Fig. 2.3. Correlation function at the critical temperature.

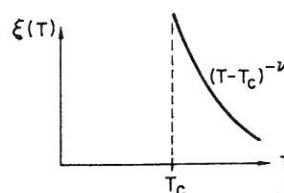


Fig. 2.4. Correlation length plotted against temperature.

One can also consider the correlation length $\xi(T)$ itself. $\xi(T)$ sets the scale for the falloff of the correlation function when T is above T_c . As shown in fig. 2.4, $\xi(T)$ approaches infinity as $T \rightarrow T_c$ from above:

$$\xi(T) \sim (T - T_c)^{-\nu} \quad (2.5)$$

$$\xi(T) \sim (T - T_c)^{-\nu} \quad \nu = 1/\nu_c \quad T = T - T_c$$

There is also a similar curve for the magnetic susceptibility,

$$\chi \equiv \left. \frac{\partial M}{\partial H} \right|_{H=0} \quad (2.6)$$

which behaves near the critical temperature as

$$\left[\chi \sim (T - T_c)^{-\gamma} \right] \quad (2.7)$$

Similarly, the specific heat (at constant volume) C_v is controlled near the critical temperature by the critical exponent α ,

$$\left[C_v \sim (T - T_c)^{-\alpha} \right] \quad (2.8)$$

We collect in table 2.1 both experimental and theoretical values of the various exponents. The theoretical predictions come from the mean field (Landau) theory which will be described below,

Table 2.1

Critical exponent	Experiment	Mean field theory	Three-dimensional Ising model	Two-dimensional Ising model
β	0.3–0.38	$\frac{1}{2}$	0.31	0.125
η	0–0.1	0	0.056	0.25
ν	0.6–0.7	$\frac{1}{2}$	0.64	1.0
γ	1.2–1.4	1	1.25	1.75
α	0–0.1	0	0.12	0

the two-dimensional Ising model solved exactly by Onsager, and the three-dimensional Ising model. In the three-dimensional case the critical exponents are calculated approximately using the high temperature expansion of the model carried to very high order. To obtain critical exponents Padé approximant techniques are used. These assume power behavior near the critical point for the relevant thermodynamic quantities. The data come from ferromagnets, anti-ferromagnets, liquid-gas transitions, binary alloys and superfluid helium. Typically one or two critical exponents can be measured in each system. The table has been organized to illustrate the idea (to be explained in later sections) that the mean field theory accounts for systems with more than four dimensions, and that exponents depend continuously on dimensionality below 4 dimensions. One can plot critical exponents against the dimensionality of the system and construct a smooth interpolating curve between the various dimensions. Fig. 2.5 depicts such a plot for the exponent γ . The curve has a break in its derivative at dimension four and is smooth elsewhere.

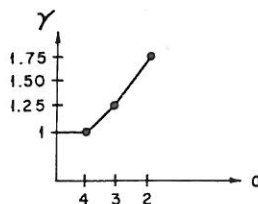


Fig. 2.5. Critical exponent γ plotted against the dimension of the physical system.

We will concentrate on the exponents η , ν and γ which have to do with the behaviour of the system at or above the critical temperature when no external field is present. We will not work in the two-phase region, although it would be more interesting; all the fundamental problems are present above T_c .

2.2. The search for analyticity

The various theories of critical phenomena are characterized by a "search for analyticity". The formulas we have written down, such as for spontaneous magnetization, are nonanalytic at the critical temperature. There is nothing really unusual about this, but it is difficult to see, starting from a fundamental formulation of the Ising model, for example, how such nonanalytic behavior arises. Recall that one calculates the thermodynamic functions from the partition function,

$$Z = \sum_{\{s\}} \exp \left\{ K \sum_n \sum_i s_n s_{n+i} \right\}, \quad K = J/kT. \quad (2.9)$$

As long as the number N of spins is finite, then Z is an analytic function of K . Such a system clearly cannot display critical behavior. However, once we take the thermodynamic limit of a spin system filling all of space, the analyticity is no longer guaranteed. As the volume V of the system tends to infinity, various sums will also become unbounded. The nonanalytic behavior we wish to discover here, however, is usually masked by these large volume effects. Thus, while nonanalyticity is allowed in principle, it has been very difficult to obtain the correct nonanalytic behavior in practice from expressions such as (2.9). Because of this one tries to find a description of critical behavior in terms of analytic functions, with the hope that the analytic functions could be obtained more easily by practical methods. The various theories can be characterized by their choice of analytic functions with which to describe critical phenomena. This is the "search for analyticity". We will discuss this problem at two levels: First, the mean field theory which makes a very simple theory of the analyticity; and the second, the renormalization group itself.

2.2.1. Mean field theory

A result of mean field theory is that there is a thermodynamic function which is analytic in its variables at T_c . Namely, the magnetic field H is an analytic function of the magnetization (per spin) M and the temperature T . This suggests Landau's hypothesis [102] that the thermodynamic properties of the system should be derivable from a free energy G which is an analytic function of M and T . Then H would be given by,

$$H = \frac{\partial}{\partial M} G(M, T). \quad (2.10)$$

What are the consequences of this approach? Near T_c where M is small we can write a Taylor series for G in powers of M ,

$$G(M, T) = G_c(T) + r(T)M^2 + u(T)M^4 + \dots \quad (2.11)$$

where M is the magnetization per unit spin. Only even powers of M appear in the expansion because of the up-down symmetry of the Ising lattice. The magnetic field is given by

$$H = 2r(T)M + 4u(T)M^3 + \dots \quad (2.12)$$

If we just consider the M and M^3 terms and assume that $u > 0$ (such that the magnetic field grows with M when M is large) we are left with two possible phenomena. If $r > 0$, then G will

Free energy is assumed to be an analytic function.

have one minimum as a function of M (fig. 2.6a). But if $r < 0$, G can have two extra minima corresponding to nonvanishing magnetizations (fig. 2.6b) and hence three local extrema corresponding to $H = 0$. However, the system will choose a state of minimum free energy which corresponds, according to fig. 2.6b, to a state of nonvanishing magnetization. We see that the possibility $r > 0$ corresponds to $T > T_c$ where there is no spontaneous magnetization, while $r < 0$ means $T < T_c$.

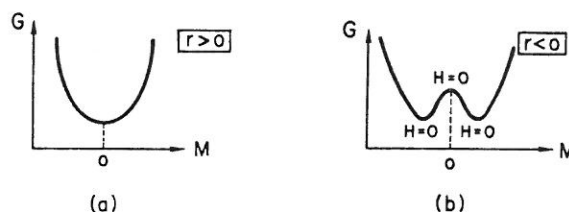


Fig. 2.6. Plots of free energy versus magnetization for the possible choices of the parameter r .

Continuing with these analyticity assumptions we can obtain some of the critical exponents. Namely, it is clear that $r(T_c) = 0$ while $u(T_c)$ will attain some particular value. The next stage is to assume that for T near T_c ,

$$r(T) \propto (T - T_c). \quad (2.13)$$

Then at $H = 0$ it follows from (2.12) that,

$$M = \sqrt{-\frac{r(T)}{2u(T)}} \sim (T_c - T)^{1/2} \quad (2.14)$$

from which we identify

$$\left[\beta = \frac{1}{2} \right] \quad (2.15)$$

as the first critical exponent. If $T > T_c$, then near zero magnetization,

$$M = H/2r(T) \quad (2.16)$$

from which we identify the susceptibility

$$\chi = 1/2r(T) \sim (T - T_c)^{-1} \quad (2.17)$$

and another critical exponent, γ , is obtained:

$$\left[\gamma = 1. \right] \quad (2.18)$$

The values of the exponents ν and η can be obtained by extending these ideas to the case of space-dependent fields. One obtains $\nu = \frac{1}{2}$ and $\eta = 0$.

Unfortunately, as we have seen from table 2.1, the mean field theory does not predict the critical exponents very well. For example, accurate experiments with liquids, with $(T - T_c)/T_c$ as small as 10^{-5} , give $\beta = 0.35 \pm 0.01$ [96] as opposed to the mean field prediction of 0.5.

However, mean field theory provides a very simple picture of critical behavior. And, although its predictions are not very good, they are not very bad, either. In our later description of critical exponents the mean field theory results will act as zeroth order approximations. Then a systematic expansion in the dimension of the system will allow us to obtain more accurate results.

2.2.2. Kadanoff theory

Now we turn to the second level of discussion. We will discuss the Kadanoff picture of the critical point [103]. The basic results of the Kadanoff picture are relations among critical exponents which had been guessed earlier by equally heuristic arguments (see [95–101]). These relations will not be discussed here. What will be discussed is where one has analyticity, and how the critical singularities are generated starting from analytic functions.

We have used considerable poetic license in describing Kadanoff's picture. The following paragraphs explain the spirit of his ideas but details of his work will be considerably distorted.

Kadanoff is concerned explicitly with the problem of the large correlation length near T_c . A direct calculation of critical behavior requires that one consider (at the minimum) all the spins in a volume the size of the correlation length – a hopeless task. In contrast, away from T_c where the correlation length is small, only a few spins need be considered at any one time. Standard techniques are available for such problems, such as Feynman graphs and perturbation theory.

Kadanoff had a brilliant idea which allows the hopeless problem with a large correlation length to be replaced by one with a small correlation length. The idea will be explained for a plane lattice, for simplicity. Consider a small region containing four spins, say. Because the correlation length is so long near the critical temperature, all the spins in such a little block should be strongly correlated. Kadanoff supposed that they are so well correlated that the four spins in one block have only two possible states: all spins up or all spins down. This means that the block of four spins acts like a single effective spin. Now suppose ξ is 1000 in units of the lattice spacing (fig. 2.7a). Group the spins into blocks of four spins each (fig. 2.7b). Each block is supposed to have only two spin degrees of freedom, so there is a single spin variable for each block. Therefore, one can replace the original lattice with an effective lattice where now $\xi = 500$ in units of the effective lattice spacing (fig. 2.7c). In this way the problem with $\xi = 1000$ is reduced to a problem with $\xi = 500$. Repetition of this analysis allows further reductions in ξ , to

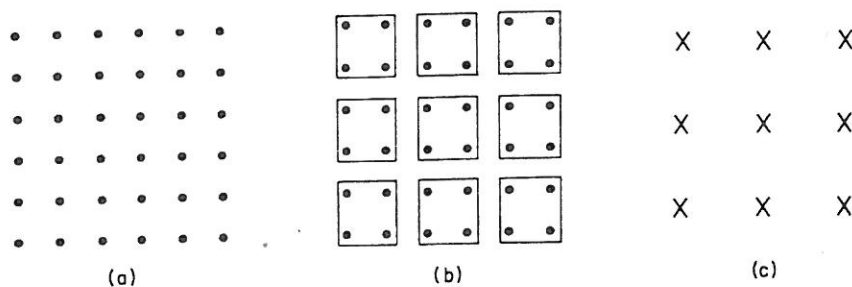


Fig. 2.7. Visualization of Kadanoff's construction of block spins. The original lattice (a) of spins at each site is divided into blocks (b) of 4 spins per block which is replaced by a new lattice (c) of "effective" spins.

250, 125, etc., until finally one has an effective theory with $\xi \sim 1$. This will be discussed further below. If the original spins have only nearest neighbor couplings, then the effective block spins also can have only nearest neighbor interactions (see later). It is convenient to define renormalized block spins such that their magnitude is ± 1 instead of ± 4 . Then the energy/ kT of the block spins is

$$\sum_{n,i} K_1 s_n^{(1)} s_{n+i}^{(1)} \quad (2.19)$$

where K_1 is a constant and $s_n^{(1)}$ is the block spin variable and n labels sites on the effective lattice

(X's in fig. 2.7c). The only practical difference between Kadanoff's effective interaction (2.19) and the original interaction $\sum_{n,i} K s_n s_{n+i}$ is the change in constants from K to K_1 . It is easy to determine the constant K_1 . There are two spin-spin interactions from the original interaction which couple the block spin $s_n^{(1)}$ to the block spin $s_{n+i}^{(1)}$. If these block spins are parallel, the coupling energy is $2K$; if they are antiparallel the coupling energy is $-2K$. So

$$[K_1 = 2K.] \quad (2.20)$$

Suppose now one solves the original interaction and obtains the correlation length $\xi(K)$ as a function of K , in units of the original lattice spacing. The correlation length for the lattice of block spins will be $\xi(K_1)$, in units of the block spin lattice spacing, since the block-spin interaction has Ising form. But now the correlation length in units of block-spin spacing must be $\frac{1}{2}$ of the original correlation length, i.e.

$$[\xi(K_1) = \frac{1}{2} \xi(K).] \quad (2.21)$$

Given that $\xi(K_1) = \frac{1}{2} \xi(K)$ whenever $K_1 = 2K$, the dependence of $\xi(K)$ on K is severely restricted. Suppose one is at the critical temperature. There is a corresponding value $K_c = J/kT_c$ for K . At K_c , ξ is infinite. But now the correlation length for the block spins must also be infinite; this means $\xi(K_1) = \infty$ which is possible only if K_1 is also K_c . So

$$K_c = K_1 = 2K_c, \quad (2.22)$$

which gives $K_c = 0$ or $K_c = \infty$, not $K_c = J/kT_c$.

This is a nonsensical result: K_c is a finite number, not 0 or ∞ . The trouble originates with the assumption that the spins in a block align exactly. Kadanoff did not actually assume this. He proposed only that the block would behave as if it had only two possible states, and therefore could be replaced by an effective block spin. However these two states would not be the states with all spins up or down. Kadanoff proposed that there would be an effective Ising interaction for the block spins, with K_1 being some function $f(K)$, but the function f would be more complicated than $2K$ obtained above: Kadanoff gave no prescription for determining $f(K)$. What Kadanoff does assume is that $f(K)$ is still an analytic function of K , even for $K = K_c$. The rationale for this is the hope that only the spins in the immediate neighborhood of the block n affect the calculation of K_1 , even though this calculation cannot be spelled out. Nonanalyticity at K_c should occur only for quantities involving the entire lattice.

The statement of the Kadanoff assumptions is that there exists an analytic function $f(K)$ such that

$$[\xi[f(K)] = \frac{1}{2} \xi(K).] \quad (2.23)$$

What does this imply for critical behavior? First one must have

$$K_c = f(K_c) \quad (2.24)$$

so that $\xi[f(K_c)]$ is infinite. Secondly, suppose K is near K_c . Approximately, one can write

$$f(K) = f(K_c) + \lambda(K - K_c) \quad (2.25)$$

where $\lambda = df/dK$ for $K = K_c$. This means

$$f(K) - K_c = \lambda(K - K_c). \quad (2.26)$$

Now suppose $\xi(K)$ behaves as $(K - K_c)^{-\nu}$ for K near K_c . Then one must have

$$\frac{\xi[f(K)]}{\xi(K)} = \left\{ \frac{f(K) - K_c}{K - K_c} \right\}^{-\nu} \quad (2.27)$$

if K is near K_c . But from (2.23) and (2.26), this equation reduces to

$$\frac{1}{2} = [\lambda]^{-\nu}. \quad (2.28)$$

If one could determine $f'(K_c)$, one could calculate ν from this equation:

$$\nu = \ln 2 / \ln \lambda. \quad (2.29)$$

For example, if $f(K)$ were the explicit form $2K$ obtained earlier, then λ would be 2 and ν would be 1.

Equation (2.26) does not force $\xi(K)$ to have power law behavior for $K \rightarrow K_c$; the most general form for $\xi(K)$ consistent with (2.23) and (2.26) is

$$\xi(K) = (K - K_c)^{-\nu} F[\ln(K - K_c)], \quad (2.30)$$

where $F(x)$ is a periodic function of x with period $\ln \lambda$. The reason for the periodicity is that if $\xi(K)$ is known, then $\xi(K_1)$ is also known where $K_1 - K_c = \lambda(K - K_c)$; changing $K - K_c$ by a factor λ is equivalent to translating $\ln(K - K_c)$ by an amount $\ln \lambda$. Because F is periodic, the behavior of $\xi(K)$ is not qualitatively different from $(K - K_c)^{-\nu}$.

To determine $\xi(K)$ precisely one can imagine proceeding as follows. Suppose $\xi(K)$ is known for $K > 2K_c$. For K above $2K_c$, ξ should not be very large and should be relatively easy to calculate. Now suppose K lies in the range $K_c < K < 2K_c$. Construct a sequence of effective constants K_1, K_2 , etc. satisfying

$$K_{n+1} = f(K_n) \quad (2.31)$$

with $K_1 = f(K)$. Then Kadanoff's formula implies

$$\xi(K) = 2^n \xi(K_n). \quad (2.32)$$

This means in particular that no matter what value K takes ($> K_c$ itself), one can choose n so large that $\xi(K_n)$ is small enough that $K_n > 2K_c$. Then $\xi(K_n)$ is known and $\xi(K)$ is determined by (2.32). This calculation is of course possible only if the function $f(K)$ is known.

What is important in Kadanoff's analysis is the idea that starting from an analytic function $f(K)$ one generates a nonanalytic behavior for $\xi(K)$ at the point K_c for which $f(K_c) = K_c$. Furthermore one does *not* get an explicit value for ν independent of the function $f(K)$; to determine ν one must know the function $f(K)$. Hence ν need not be the mean field value $\frac{1}{2}$. In fact ν can be irrational, contrary to the dreams of some statistical mechanicians.

In the following, Kadanoff's idea that there exist effective block-spin interactions with coupling constants analytic in T will be realized in various forms, with all functions given explicitly. Critical exponents such as ν will be computed explicitly.

3. Trivial example of the renormalization group: The Gaussian model

In this section we will begin to make Kadanoff's intuitive ideas quantitative. The relationship between the block spin interaction and the original interaction will be worked out and a critical exponent (ν) will be computed for a trivial model — the Gaussian model [104] — to illustrate the ideas involved. In this case the exponent ν has the mean field value $\frac{1}{2}$.

The Gaussian model can be obtained by modifying the Ising model. First one writes the partition function in terms of integrals. Namely,

$$Z = \prod_m \int_{-\infty}^{\infty} ds_m 2\delta(s_m^2 - 1) \exp \left\{ K \sum_n \sum_i s_n s_{n+i} \right\}. \quad (3.1)$$

This is a trivial rewriting of the original partition function. Now imagine smoothing the delta function (fig. 3.1a) to a smooth distribution around $s_n = \pm 1$ (fig. 3.1b), to a smooth Gaussian

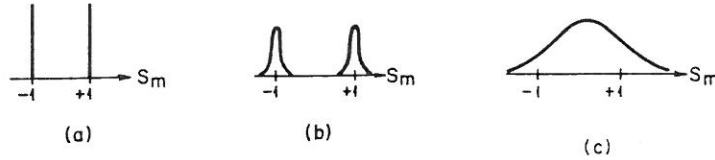


Fig. 3.1. The transition from the Ising to the Gaussian model. The Ising model (a) has spin up or spin down at each lattice site. Model (b) has spin variables which peak about the Ising values. The Gaussian model (c) has spin variables at each site with smooth, Gaussian distributions about zero

(fig. 3.1c). Of course fig. 3.1c bears little resemblance to the original Ising model. If we make this replacement in the partition function anyway,

$$Z = \prod_m \int_{-\infty}^{\infty} ds_m \exp \left\{ -\frac{1}{2} b s_m^2 \right\} \exp \left\{ K \sum_n \sum_i s_n s_{n+i} \right\} \quad (3.2)$$

where b is an arbitrary constant. This formula defines the Gaussian model [104]. Later we will allow the generalization,

$$\exp \left\{ -\frac{1}{2} b s_n^2 \right\} \rightarrow \exp \left\{ -\frac{1}{2} b s_n^2 - u s_n^4 \right\} \quad (3.3)$$

where u is a positive number. The case u near 1 (not small) can be discussed using the recursion formula to be derived later. In this case the smooth model begins to approach the real Ising model. Finally, if $u \rightarrow \infty$ and $b \rightarrow -\infty$, with $b = -4u$, the original Ising model is recovered (if one includes a constant factor $(u/\pi)^{1/2} \exp(-u)$ per spin).

Let us review some characteristics of Gaussian integrals. It is convenient to introduce matrix notation,

$$\tilde{A}s \equiv \sum_n \sum_m s_n A_{nm} s_m, \quad \tilde{\rho}s \equiv \sum_n \rho_n s_n \quad (3.4)$$

with A assumed to be symmetric for $n \leftrightarrow m$. Then, using techniques to be developed shortly, the following integral can be evaluated,

$$\prod_n \int_{-\infty}^{\infty} ds_n \exp \left\{ -\frac{1}{2} \tilde{A}s + \tilde{\rho}s \right\} = C \exp \left\{ \frac{1}{2} \tilde{\rho} A^{-1} \rho \right\} \quad (3.5)$$

where C , a normalization factor, is a function of A . (Specifically $C = (\det A)^{-1/2} (2\pi)^{N/2}$ where N is the number of sites.) Eq. (3.5) is derived by translating s_n in order to complete the square in the

integrand. (See eq. (3.21) below for an example of a translation.) From eq. (3.5) one can derive all integrals of the form

$$\prod_n \int_{-\infty}^{\infty} ds_n(s_{m_1} s_{m_2} \dots s_{m_k}) \exp\left\{-\frac{1}{2} \tilde{A} s\right\} = I(m_1 \dots m_k). \quad (3.6)$$

These integrals will be needed in section 4. The procedure is to differentiate eq. (3.5) with respect to $\rho_{m_1} \dots \rho_{m_k}$. This gives

$$\prod_n \int_{-\infty}^{\infty} ds_n(s_{m_1} \dots s_{m_k}) \exp\left\{-\frac{1}{2} \tilde{A} s + \tilde{\rho} s\right\} = \frac{\partial}{\partial \rho_{m_1}} \dots \frac{\partial}{\partial \rho_{m_k}} C \exp\left\{\frac{1}{2} \tilde{\rho} A^{-1} \rho\right\}.$$

The second step is to set $\rho_m = 0$ for all m . If k is odd the integral vanishes; if k is even one needs to consider only the term of order ρ^k from the exponential, giving

$$I(m_1, \dots, m_k) = \frac{\partial}{\partial \rho_{m_1}} \dots \frac{\partial}{\partial \rho_{m_k}} C \left\{ \frac{1}{2} \tilde{\rho} A^{-1} \rho \right\}^{k/2} / (k/2)!. \quad (3.7)$$

One now has k derivatives applied to $k/2$ factors of $(\frac{1}{2} \tilde{\rho} A^{-1} \rho)$. To illustrate the calculation of the derivatives consider the case $k = 4$. Then one has to calculate

$$\frac{\partial}{\partial \rho_{m_1}} \frac{\partial}{\partial \rho_{m_2}} \frac{\partial}{\partial \rho_{m_3}} \frac{\partial}{\partial \rho_{m_4}} \left\{ \frac{1}{2} \tilde{\rho} A^{-1} \rho \right\}^2 / 2!.$$

Differentiating yields a number of terms such as $(A^{-1})_{m_1 m_2} (A^{-1})_{m_3 m_4} / 8$. The total number of such terms is 24, namely the total number of ways of pairing each derivative with a ρ or $\tilde{\rho}$. However, many of these terms are identical to the one just cited. Due to the symmetry of A^{-1} , the term $(A^{-1})_{m_2 m_1} (A^{-1})_{m_3 m_4} / 8$ is identical. Obviously the term $(A^{-1})_{m_3 m_4} (A^{-1})_{m_1 m_2} / 8$ is identical. Combining all identical terms one finds

$$I(m_1, \dots, m_4) = C \{ (A^{-1})_{m_1 m_2} (A^{-1})_{m_3 m_4} + (A^{-1})_{m_1 m_3} (A^{-1})_{m_2 m_4} + (A^{-1})_{m_1 m_4} (A^{-1})_{m_2 m_3} \}. \quad (3.8)$$

The general rule for any k is constructed similarly. The result is as follows. Let $\overline{s_{m_1} s_{m_2}}$ stand for $(A^{-1})_{m_1 m_2}$. One calls $\overline{s_{m_1} s_{m_2}}$ a "contraction". Then $I(m_1, \dots, m_k)$ is the sum of all possible ways of contracting $s_{m_1} s_{m_2} \dots s_{m_k}$ in pairs, such that all s_{m_i} are in a contraction. For example

$$I(m_1, m_2, m_3, m_4) = C \{ \overline{s_{m_1} s_{m_2}} \overline{s_{m_3} s_{m_4}} + \overline{s_{m_1} s_{m_3}} \overline{s_{m_2} s_{m_4}} + \overline{s_{m_1} s_{m_4}} \overline{s_{m_2} s_{m_3}} \} \quad (3.9)$$

which is the same as eq. (3.8). $I(m_1 \dots m_6)$ consists of $\overline{s_{m_1} s_{m_2}} \overline{s_{m_3} s_{m_4}} \overline{s_{m_5} s_{m_6}}$ plus 14 other terms. Note that it is not necessary for the m_i to be distinct. For example, suppose one is dealing with trivial 1×1 matrices so that the m_i are necessarily all equal to 1. Then the integral (3.6) of s_1^4 is $3CA_{11}^{-2}$, the integral of s_1^6 is $15CA_{11}^{-3}$, etc., in agreement with standard formulae. The rule for $I(m_1, \dots, m_k)$ is analogous to Wick's theorem in quantum field theory [105].

In the cases of interest $A_{n,m}$ only depends on the difference $n - m$ and the inverse matrix $(A^{-1})_{n,m}$ can be computed by Fourier transforms: write

$$A_{n-m} = \int_q \exp\{iq \cdot (n-m)\} A(q) \quad (3.10)$$

where

$$\int_q \equiv \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} dq_1 \dots \int_{-\pi}^{\pi} dq_d.$$

The inverse matrix is obtained by taking the transform of the ordinary inverse of $A(q)$,

$$(A^{-1})_{n-m} = \int_q \exp\{iq \cdot (n-m)\} A^{-1}(q) \quad (3.11)$$

since A is diagonal in the q representation.

Return now to the Gaussian model. We can rewrite the expression of interest,

$$K \sum_n \sum_i s_n s_{n+i} - \frac{1}{2} b \sum_n s_n^2 = -\frac{1}{2} K \sum_n \sum_i (s_{n+i} - s_n)^2 - \left(\frac{1}{2} b - dK\right) \sum_n s_n^2 \quad (3.12)$$

where we are supposing for convenience that the sum over i runs only over positive values. d is the dimensionality of the lattice. When written in terms of σ_q , $\sigma_q = \sum_n \exp\{-iq \cdot n\} s_n$, (3.12) becomes

$$-\frac{1}{2} \int_q \left\{ K \sum_i |\exp(iq_i) - 1|^2 + \tilde{r} \right\} \sigma_q \sigma_{-q} \quad (3.13)$$

with

$$\tilde{r} = b - 2dK \quad (3.14)$$

where q_i is the i th component of q . There are now three further changes that will be made on the model. The first change is to replace $|\exp(iq_i) - 1|^2$ by its form for small q , namely q_i^2 . The second is to change the range of integration over q to be $0 < |q| < 1$ instead of $-\pi < q_i < \pi$. The third change is to renormalize the spins so that $K = 1$. So finally, the interaction is

$$-\frac{1}{2} \int_q (q^2 + r) \sigma_q \sigma_{-q} \quad (3.15)$$

with \int_q being $(2\pi)^{-d} \int d^d q$ for $0 < |q| < 1$ and $r = \tilde{r}/K$.

The second change causes a conceptual difficulty. With $0 < |q| < 1$, one cannot relate the functional variable σ_q to the ordinary variables s_n . Therefore, one is forced to define the partition function as a functional integral over σ_q rather than ordinary integrals over the s_n . However, for present purposes this is not a difficulty. The reason is that the method of translating the integration variable is valid for functional integrals as well as ordinary integrals, and is as easy to use. The reason one can translate variables in a functional integral is that a functional integral is generally defined as a limit of ordinary integrals, each of which can be translated. For example, the functional integral over σ_q can be described as a limit of ordinary integrals over variables σ_{q_m} in the limit that the spacing of the points q_m goes to zero (see, e.g., [106]).

The replacement of $|\exp(iq_i) - 1|^2$ by q_i^2 is not an essential change in the model because we will ultimately be interested only in the long wavelength behavior of the model which comes from the part with small q . Likewise the change in the range of $|q|$ is inessential.

The correlation function of the model will now be calculated. It is appropriate to introduce a spin field $s(x)$:

$$s(x) = \int_q \exp(iq \cdot x) \sigma_q \quad (0 < |q| < 1) \quad (3.16)$$

to replace s_n . Then the correlation function is

$$\Gamma(x) = \int_{\sigma} s(x) s(0) \exp\left\{-\frac{1}{2} \int_q \sigma_q \sigma_{-q} (q^2 + r)\right\} / Z \quad (3.17)$$

where \int_{σ} denotes the functional integral over functions $\sigma(q)$ and Z is the functional integral of the exponential alone. The result for $\Gamma(x)$ is

$$\Gamma(x) = \int_q \exp(iq \cdot x) \frac{1}{q^2 + r} = \int_q \exp(iq \cdot x) \Gamma_q. \quad (3.18)$$

This result will now be derived. One has

$$\Gamma(x) = \int_q \int_{q_1} \exp(iq \cdot x) \int_{\sigma} \sigma_q \sigma_{q_1} \exp\left\{-\frac{1}{2} \int_q (q^2 + r) \sigma_q \sigma_{-q}\right\} / Z. \quad (3.19)$$

The functional integral is computed from the generating function

$$Z(j) = \frac{1}{Z} \int_{\sigma} \exp\left\{-\frac{1}{2} \int_q (q^2 + r) \sigma_q \sigma_{-q} + \int_q j_{-q} \sigma_q\right\}. \quad (3.20)$$

Translating variables means in this case writing

$$\sigma_q = \frac{j_q}{q^2 + r} + \sigma'_q. \quad (3.21)$$

In terms of σ'_q ,

$$Z(j) = \frac{1}{Z} \int_{\sigma'} \exp\left\{-\frac{1}{2} \int_q (q^2 + r) \sigma'_q \sigma'_{-q} + \frac{1}{2} \int_q j_q j_{-q} \left(\frac{1}{q^2 + r}\right)\right\}. \quad (3.22)$$

The functional integral is proportional to Z , giving

$$Z(j) = \exp\left\{\frac{1}{2} \int_q j_q j_{-q} \left(\frac{1}{q^2 + r}\right)\right\}. \quad (3.23)$$

Expanding (3.20) and (3.23) to second order in j one sees that

$$\frac{1}{2} \int_q \int_{q_1} j_{-q} j_{-q_1} \int_{\sigma} \sigma_q \sigma_{q_1} \exp\left\{-\frac{1}{2} \int_q (q^2 + r) \sigma_q \sigma_{-q}\right\} / Z = \frac{1}{2} \int_q j_q j_{-q} \frac{1}{q^2 + r}. \quad (3.24)$$

This equation must be an identity in the function j , which means

$$\int_{\sigma} \sigma_q \sigma_{q_1} \exp\left\{-\frac{1}{2} \int_q (q^2 + r) \sigma_q \sigma_{-q}\right\} / Z = \delta(q + q_1) / (q^2 + r) \quad (3.25)$$

where $\delta(q + q_1)$ is shorthand for $(2\pi)^d \delta^d(q + q_1)$. Substitution of this formula in (3.19) gives the result (3.18) for $\Gamma(x)$.

The correlation length ξ is customarily defined in terms of the behavior of $\Gamma(x)$ for $|x| \rightarrow \infty$. If the range of q were infinite, the behavior of $\Gamma(x)$ for large x would be governed by the singularity in $(q^2 + r)^{-1}$ for $q = \pm i\sqrt{r}$, resulting in

$$\Gamma(x) \propto \exp(-\sqrt{r}|x|) \quad (3.26)$$

for large $|x|$ (apart from a power of $|x|$ in front of the exponential). From this one gets $\xi = 1/\sqrt{r}$. Unfortunately, the sharp boundary at $|q| = 1$ leads to another term in $\Gamma(x)$ for large x , behaving as $\cos(|x|)$. This latter term is an artifact of the model; it can be removed by averaging over a large but finite region in x space. When ξ is large (r small) the term $\exp(-\sqrt{r}|x|)$ will not be

of affected by this averaging. However a more convenient procedure for our purposes is to introduce an alternative definition of the correlation length. The correlation length as originally defined is determined by the leading singularity of Γ_q : if the leading singularity of Γ_q is at $q = \pm i q_s$ then ξ is $1/q_s$. The location of this singularity can be determined roughly by comparing the derivative $d\Gamma_q/dq^2$ with Γ_q itself when $q = 0$, namely,

$$\xi^2 \propto - \frac{d\Gamma_q/dq^2}{\Gamma_q} \bigg|_{q=0} \quad (3.27)$$

Equivalently,

$$\xi^2 = \int x^2 \Gamma(x) d^d x / \int \Gamma(x) d^d x.$$

With this definition, ξ is called the "effective range of correlation" [107]. For $\Gamma_q = 1/(q^2 + r)$ this formula gives

$$\xi^2 \propto 1/r \quad (3.28)$$

and hence $\xi \propto 1/\sqrt{r}$ as before.

In the Gaussian model r is linear in the temperature T . To be precise, with $K = J/kT$ (from eq. (2.9)) one has

$$r = (bkT - 2dJ)/J \quad (3.29)$$

(see eqs. (3.14) and (3.15)).

The critical point corresponds to $r = 0$ and $\xi = \infty$; clearly $\xi \propto (T - T_c)^{-1/2}$ for $T > T_c$. Hence, the Gaussian model gives the mean field value $\frac{1}{2}$ for ν . Also this gives $kT_c = 2dJ/b$. The critical value of K is $K_c = b/2d$.

An exact formulation of the renormalization group can be defined for the Gaussian model; it illustrates the ideas that will be applied later to non-trivial models.

As a first step in the renormalization group transformation, let us integrate out the spin components σ_q with $\frac{1}{2} < |q| < 1$, leaving unintegrated the components σ_q with $0 < |q| < \frac{1}{2}$. In other words one integrates out the rapidly fluctuating parts of the spin field $s(x)$, leaving unintegrated the slowly fluctuating parts. This is a way of realizing in spirit Kadanoff's idea of producing an effective interaction involving only block spin variables; namely, one integrates out the other variables orthogonal to the block spin variables. In our case one thinks of the long wavelength spin components σ_q for $|q| < \frac{1}{2}$ as analogous to Kadanoff's block spin variables.

The integration of σ_q for $\frac{1}{2} < |q| < 1$ will be done in the functional integral for the partition function, as opposed to the functional integral for the correlation function. However the same calculation applies also to the correlation function integral if one considers the functional integral only for Γ_{q_1} for $|q_1| < \frac{1}{2}$, where

$$\delta(q_1 + q_2) \Gamma_{q_1} = \int \sigma_{q_1} \sigma_{q_2} \exp \left\{ -\frac{1}{2} \int_q (q^2 + r) \sigma_q \sigma_{-q} \right\} / Z. \quad (3.30)$$

If $|q_1|$ and $|q_2|$ are both less than $\frac{1}{2}$, then the explicit σ_{q_1} and σ_{q_2} terms are not involved in the calculation of integrals for $\frac{1}{2} < |q| < 1$.

The calculation of the first step defined above is trivial. The reason is that there is no coupling between σ_q for $|q| > \frac{1}{2}$ and σ_q for $|q| < \frac{1}{2}$. Hence the only result of the functional integral over σ_q with $|q| > \frac{1}{2}$ is a constant which multiplies $\exp \left\{ -\frac{1}{2} \int'_q (q^2 + r) \sigma_q \sigma_{-q} \right\}$ where \int'_q means an integral

over $0 < |q| < \frac{1}{2}$ only. Since this multiplicative constant cancels in the ratio of functional integrals defining Γ_q , the constant can be ignored. Thus the effective interaction produced by eliminating the high momentum spin components is simply $-\frac{1}{2} \int_q' (q'^2 + r) \sigma_q \sigma_{-q}$.

The second step in constructing the renormalization group transformation is to perform some scale changes designed to make the effective interaction look as much like the original interaction as possible. Two scale changes will be introduced. A scale change in the momentum, namely $q' = 2q$, is made so that the new momentum variable q' has the range $0 < |q'| < 1$ like the original momentum q in the original interaction. The second scale change is to scale the spin variable itself:

$$\sigma_q = \xi \sigma_{q'} = \xi \sigma_{2q}. \quad (3.31)$$

The scale factor ξ will be determined later. In terms of the new spin variable σ' , the effective interaction is

$$\mathcal{H}' = -\frac{1}{2} (\xi^2 2^{-d}) \int_{q'} (q'^2/4 + r) \sigma_{q'}' \sigma_{-q'}'. \quad (3.32)$$

Now choose ξ so that the q'^2 term has the same coefficient as the q^2 term of the original interaction; this means

$$\xi = 2^{1+d/2} \quad (3.33)$$

and

$$\mathcal{H}' = -\frac{1}{2} \int_{q'} (q'^2 + r') \sigma_{q'}' \sigma_{-q'}' \quad (3.34)$$

with

$$r' = 4r. \quad (3.35)$$

The transformation that takes

$$\mathcal{H} = -\frac{1}{2} \int_q (q^2 + r) \sigma_q \sigma_{-q} \quad (3.36)$$

into \mathcal{H}' is an example of a renormalization group transformation.

The two scale changes introduced above are analogous to scale changes that occur in the Kadanoff picture. When one discusses the block spin interaction, one uses the spacing of blocks as a length scale instead of the original lattice spacing. This is analogous to the momentum scale change introduced here. Secondly, the magnitude of the block spins was rescaled so it would be ± 1 like the original spins; analogously, the spins σ_q are rescaled, although now the purpose is to make a particular term in the interaction have the same form as in the original interaction.

What is \mathcal{H}' good for? It can be used to compute the correlation function Γ_q , provided $|q| < \frac{1}{2}$. The correlation function naturally defined for \mathcal{H}' is a correlation function involving σ' instead of σ , namely,

$$\delta(q'_1 + q'_2) \Gamma_{q'_1} = \int_{\sigma'} \sigma_{q'_1}' \sigma_{q'_2}' \exp \{\mathcal{H}'\} / Z'. \quad (3.37)$$

The scaling relation between σ and σ' gives

$$\delta(q_1 + q_2) \Gamma_{q_1} = \xi^2 \Gamma_{2q_1}' \delta(2q_1 + 2q_2). \quad (3.38)$$

Now $\delta(2q) = 2^{-d} \delta(q)$ so

$$\Gamma_q = \xi^2 2^{-d} \Gamma'_{2q} = 4 \Gamma'_{2q}. \quad (3.39)$$

Thus, if one computes Γ'_q from \mathcal{H}' , one can reconstruct $\Gamma_{q/2}$ from the scaling formula.

Now look specifically at the correlation length. The correlation length for \mathcal{H}' is (using the modified definition),

$$\xi' = 1/\sqrt{r'}. \quad (3.40)$$

Because of the change in momentum scale in defining \mathcal{H}' , ξ' will be different from ξ . Since momenta were doubled in the transformation from \mathcal{H} to \mathcal{H}' , lengths will be halved, and therefore ξ' should be $\xi/2$. This is indeed the case, since

$$\xi' = (r')^{-1/2} = (4r)^{-1/2} = \xi/2. \quad (3.41)$$

Here we have an explicit example of the relation between ξ' calculated from the effective interaction \mathcal{H}' and ξ calculated for the original interaction. There are three relations involved. First there is the relation between ξ' and ξ : $\xi' = \frac{1}{2} \xi$. This relation is a consequence of the explicit change of scale in momentum introduced in the renormalization group transformation. This relation will be true of all models so long as we stick to the program of integrating out the momentum components σ_q with $|q| > \frac{1}{2}$ and then rescale the remaining momenta. The second relation is $r' = 4r$, the relation between the constants appearing in \mathcal{H} and \mathcal{H}' . This relation is special to the Gaussian model. However, note that r' is an analytic function of r , even at the critical point. This analyticity will be true of all subsequent examples. Finally there is the dependence of ξ on r : $\xi = 1/\sqrt{r}$. Since \mathcal{H}' has the same form as \mathcal{H} , the new correlation length ξ' has the same dependence on r' : $\xi' = 1/\sqrt{r'}$. These relations are special to the Gaussian model; but it will always be true that if $\xi = X(r)$ then $\xi' = X(r')$ with the same function X .

The critical singularity of ξ can be calculated by the method of section 2. The variable r substitutes for the variable K of section 2. The function $f(K)$ of section 2 is now known explicitly: $r' = f(r) = 4r$. The critical value of r is $r_c = 0$, which satisfies $f(r_c) = r_c$. The parameter λ defined in section 2 is $df/dr = 4$. Thus, from eq. (2.29)

$$\nu = \ln 2 / \ln \lambda = \frac{1}{2}. \quad (3.42)$$

In this case $\xi(r)$ must behave as $\xi(r) \propto (r - r_c)^{-1/2}$ without the extra periodic function $F[\ln(r - r_c)]$ of section 2. The reason is that the period of F is determined by the fact that ξ changes by a factor 2 [$\xi(r') = \frac{1}{2} \xi(r)$]. One could have chosen to make ξ change by an arbitrary factor s , by integrating σ_q for $s^{-1} < |q| < 1$ instead of $\frac{1}{2} < |q| < 1$. The result would still be $\nu = \frac{1}{2}$ but the extra function would have period $\ln s$ instead of $\ln 2$. This gives a contradiction unless F is a constant.

Thus, if $f(r)$ is known, one can determine both the critical point $r_c = 0$ ($K_c = b/2d$) and the critical singularity of ξ .

One starts with an analytic formula for $r' = 4r$ and obtains a nonanalytic form for ξ . Why? The reason is that as $r \rightarrow 0$, r' comes close to r but ξ' cannot come close to ξ , since ξ' must be $\xi/2$. It is this conflict that forces ξ to be singular when $r' = r = 0$.

4. The s^4 model

In this section we will generalize the discussion of the Gaussian model. In particular, we will add to the interaction a small quartic term. The renormalization problem now becomes nontrivial