

schr. Phys. 41 (1993) 4, 307–420

Fortschr. ~~Phys.~~ Phys.

Techniques for the Calculation of Electroweak Radiative Corrections at the One-Loop Level and Results for W -physics at LEP 200

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Abstract

We review the techniques necessary for the calculation of virtual electroweak and soft photonic corrections at the one-loop level. In particular we describe renormalization, calculation of one-loop integrals and evaluation of one-loop Feynman amplitudes. We summarize many explicit results of general relevance. We give the Feynman rules and the explicit form of the counterterms of the electroweak standard model, we list analytical expressions for scalar one-loop integrals and reduction of tensor integrals, we present the decomposition of the invariant matrix element for processes with two external fermions and we give the analytic form of soft photonic corrections. These techniques are applied to physical processes with external W -bosons. We present the full set of analytical formulae and the corresponding numerical results for the decay width of the W -boson and the top quark. We discuss the cross section for the production of W -bosons in e^+e^- -annihilation including all $O(\alpha)$ radiative corrections and finite width effects. Improved Born approximations for these processes are given.

Contents

1. Introduction	309
2. The Glashow-Salam-Weinberg Model	311
2.1. The Yang-Mills-part	312
2.2. The Higgs part	312
2.3. Fermionic Part	313
2.4. Physical fields and parameters	314
2.5. Quantization	316
3. Renormalization	316
3.1. Renormalization constants and counterterms	318
3.2. Renormalization conditions	319
3.3. Explicit form of renormalization constants	323

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4. One-Loop Integrals	326
4.1. Definitions	327
4.2. Reduction of tensor integrals to scalar integrals	329
4.3. Scalar one-loop integrals for $N \leq 4$	332
4.3.1. Scalar one-point function	332
4.3.2. Scalar two-point function	332
4.3.3. Scalar three-point function	333
4.3.4. Scalar four-point function	334
4.4. Reduction of scalar and tensor integrals for vanishing Gram determinant	337
4.4.1. Reduction of five-point functions	337
4.4.2. Reduction of the scalar N-point function for zero Gram determinant	339
4.5. UV-divergent parts of tensor integrals	340
5. Standard Matrix Elements	340
5.1. Definition	340
5.2. Standard matrix elements for processes with two external fermions	342
5.2.1. $S \rightarrow F\bar{F}$	343
5.2.2. $V \rightarrow F\bar{F}$	343
5.2.3. $F\bar{F} \rightarrow SS$	343
5.2.4. $F\bar{F} \rightarrow SV$	343
5.2.5. $F\bar{F} \rightarrow VV$	344
5.3. Calculation of standard matrix elements	345
6. Calculation of One-Loop Amplitudes	347
6.1. Algebraic reduction of Feynman diagrams	347
6.2. Generic Feynman diagrams	350
6.3. The decay $W \rightarrow f_i \bar{f}_j'$ for massless fermions	350
6.4. Computeralgebraic calculation of one-loop diagrams	353
7. Soft Photon Bremsstrahlung	353
7.1. Soft photon approximation	354
7.2. Soft photon cross section	354
8. Input Parameters and Leading Higher Order Contributions	358
8.1. Input parameters	358
8.2. Leading higher order contributions	361
8.2.1. Leading logarithms from light fermion masses	361
8.2.2. Leading m_f^2 contributions	362
8.2.3. Recipes for leading universal corrections	363
9. The Width of the W-Boson	363
9.1. Lowest order	364
9.2. Electroweak virtual corrections	365
9.3. Photon bremsstrahlung	369
9.4. QCD corrections	370
9.5. Results and discussion	371
10. The Top Width	375
10.1. Notation and lowest order decay width	376
10.2. Virtual corrections	377
10.3. Bremsstrahlung	377
10.4. Results and discussion	378

11. W -Pair Pro

11.1. Notati

11.2. Born c

11.3. Virtua

11.3.1

11.3.2

11.3.3

11.3.4

11.4. Hard

11.4.1

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11.5. Finite

12. Conclusion**A. Feynman Ri****B. Self Energie****C. Vertex Form****D. Bremsstrahlh****1. Intro**

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 $\gamma W^+ W^-$ and
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 10^4 W -pairs a
independent m
can be reache

326	11. <i>W</i> -Pair Production	381
327	11.1. Notation and amplitudes	382
329	11.2. Born cross section	383
332	11.3. Virtual and soft photonic corrections	388
332	11.3.1 Virtual corrections	388
332	11.3.2 Soft photonic corrections	388
333	11.3.3 Leading weak corrections	390
334	11.3.4 Numerical results for the virtual and soft photonic corrections	391
337	11.4. Hard photon bremsstrahlung	394
337	11.4.1 Complete calculations	394
339	11.4.2 Leading logarithmic approximation	394
340	11.4.3 Numerical results	395
340	11.5. Finite widths effects	397
340	12. Conclusion	400
342	A. Feynman Rules	401
343	B. Self Energies	409
343	C. Vertex Formfactors	412
343	D. Bremsstrahlung Integrals	416

1. Introduction

All known experimental facts about the electroweak interaction are in agreement with the GLASHOW-SALAM-WEINBERG (GSW) model [1, 2, 3, 4]. Therefore, this theory is called the standard model (SM) of electroweak physics. Despite its extraordinary experimental success it is by no means tested in its full scope. Many more experimental and theoretical efforts are needed for its further confirmation.

An important step in this direction is provided by the e^+e^- colliders SLC and LEP100 which started a new era of precision experiments. The first important results from these experiments were the determination of the number of light neutrinos and the precise measurement of the mass of the neutral weak gauge boson, the Z-boson [5]. Furthermore the total and partial widths of the Z-boson and various on-resonance asymmetries have been determined and will be measured with increasing accuracy. These experiments will uniquely allow to study in great detail all the properties of the Z-boson and its couplings to fermions.

There are, however, ingredients of the electroweak SM, which are not directly accessible at SLC and LEP100. The most important one is probably the gauge boson self-interaction which is crucial for the nonabelian structure of the GSW model. It will be directly tested for the first time at LEP200, the upgraded version of LEP. There the center of mass energy will be high enough to produce pairs of charged weak gauge bosons, the *W*-bosons, such that one can study the reaction $e^+e^- \rightarrow W^+W^-$ in great detail. It will allow the investigation of the nonabelian three-gauge boson interactions γW^+W^- and ZW^+W^- at the classical level of the theory. Moreover, all the properties of the *W*-boson, like its mass and its total and partial widths can be measured directly there. The statistics will not be as good as on the Z-peak. One expects of the order of 10^4 *W*-pairs and thus an accuracy at the percent level. The examination of several independent methods indicates that an error of about 0.1% for the *W*-mass determination can be reached [6].

Theoretical predictions should have an accuracy comparable to or even better than the experimental errors. If the experimental precision is of the order of one percent the classical level of the theory is no longer sufficient. One is forced to take into account quantum corrections: the radiative corrections. In the case of the electroweak SM these can reach several percent. For the high precision experiments at LEP100 even the first order corrections are inadequate, one has to take into account leading higher order corrections, too.

Radiative corrections are not only compelling for the precise comparison between the theoretical predictions and the experimental results, but offer the possibility to get informations about sectors of the theory that are not directly observable. While the direct investigation of certain objects may not be possible because the available energy is too small to produce them they may affect the radiative corrections noticeably.

In the electroweak SM there are at least two such objects. The top quark, the still undiscovered constituent of the third fermion generation, and the Higgs boson, the physical remnant of the Higgs-Kibble mechanism of spontaneous symmetry breaking. Both particles seem to be too massive to be produced directly in the existing colliders. However, the high precision experiments performed so far together with the precise knowledge of the radiative corrections of the electroweak SM already allow to derive limits on the mass of the top quark within the SM [7, 5]. Since the sensitivity of radiative corrections to the mass of the Higgs boson is weaker, the restrictions on this parameter are at present only marginal [8]. The situation may improve with increasing experimental accuracy. While direct determinations of physical parameters are in general to a large extent model independent, the information extracted from radiative corrections depends on the entire structure of the underlying theory.

Finally there is a third important issue concerning radiative corrections. It is likely that the electroweak SM, despite its experimental success, is only an effective theory, the low-energy approximation of a more general structure. This would manifest itself typically in small deviations from the SM predictions. Furthermore most of the presently discussed new physics is connected with scales bigger than the experimentally accessible energies. Therefore new phenomena will show up predominantly via indirect effects rather than via direct production of new particles. In order to disentangle these small effects one has to know once again the predictions of the SM accurately and thus needs radiative corrections.

The actual evaluation of the radiative corrections is a tedious and time consuming task. It requires extensive calculations involving many different techniques, like renormalization, evaluation of loop integrals, Dirac algebra calculations, phase space integrations and so on. Fortunately the whole procedure can be organized into different independent steps. Furthermore many steps can be facilitated with the help of computer algebra [9, 10, 11, 12].

For the interesting processes at LEP100 radiative corrections have been calculated by many authors [13]. Their structure is relatively simple since the masses of the external fermions can be neglected. Calculations for gauge boson production processes at LEP200 are already more complicated because the masses of the external gauge bosons are non-negligible. Such calculations have been performed by several groups and we will give the most important results in the second part of this review. The whole complexity of one-loop corrections will show up when considering reactions where all external particles are massive like e.g. gauge boson scattering processes which may be investigated at the LHC or SSC. The calculation of radiative corrections to these processes has just started.

In the first part of this review we collect the relevant formulae and techniques necessary for the calculation of electroweak one-loop radiative corrections. Although we discuss everything in the context of the SM the presented material is – apart from the explicit form of the renormalization constants – applicable to extended models as well. In the

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second part these methods are applied to physical processes with external W -bosons. This part not only gives examples for the calculation of one-loop electroweak corrections, but also provides a survey on the status of radiative corrections for the production and the decay of W -pairs in e^+e^- annihilation. The corresponding experiments will be carried through in a few years at LEP200.

The general techniques described in this paper are restricted to the virtual part of the electroweak corrections and soft photon bremsstrahlung. We do not consider the methods appropriate for hard photon bremsstrahlung. This can be efficiently treated using spinor techniques [14] and Monte Carlo simulations [15]. Furthermore we do not touch the methods developed for calculating higher order QCD corrections.

This paper is organized as follows:

In chapter 2 we specify the Lagrangian of the electroweak SM. Chapter 3 outlines the on-shell renormalization for the physical sector of the electroweak SM and provides explicit expressions for the counter terms. All relevant formulae for the calculation of one-loop Feynman integrals are collected in chapter 4. In chapter 5 we introduce the standard matrix elements, a concept which allows to present the results for one-loop diagrams in a systematic and simple way. In chapter 6 we show how everything is put together in the actual calculation of one-loop amplitudes and provide first simple examples. The relevant formulae for the calculation of the soft photon corrections are summarized in chapter 7. Chapter 8 serves to define our input parameters and the way of resumming higher order corrections.

The remaining chapters are devoted to applications. In chapter 9 we give results for the width of the W -boson, in chapter 10 for the width of the top quark. Finally the radiative corrections to the production of W -pairs in e^+e^- annihilation are discussed in chapter 11.

The appendices contain the Feynman rules of the electroweak SM, the explicit expressions for the self energies of the physical particles and the vertex functions as well as the bremsstrahlung integrals relevant for the W -boson and top quark decay width.

2. The Glashow-Salam-Weinberg Model

The Glashow-Salam-Weinberg (GSW) model of the electroweak interaction has been proposed by GLASHOW [1], WEINBERG [2], and SALAM [3] for leptons and extended to the hadronic degrees of freedom by GLASHOW, ILIOPOULOS and MAIANI [4]. It is the presently most comprehensive formulation of a theory of the unified electroweak interaction: theoretically consistent and in agreement with all experimentally known phenomena of electroweak origin. For energies that are small compared to the electroweak scale it reproduces quantum electrodynamics and the Fermi model, which already accomplished a good description of the electromagnetic and weak interactions at low energies. It is minimal in the sense that it contains the smallest number of degrees of freedom necessary to describe the known experimental facts.

The electroweak standard model (SM) is a nonabelian gauge theory based on the non-simple group $SU(2)_W \times U(1)_Y$. From experiment we know that three out of the four associated gauge bosons have to be massive. This is implemented via the Higgs-Kibble mechanism [16]. By introducing a scalar field with nonvanishing vacuum expectation value the $SU(2)_W \times U(1)_Y$ gauge symmetry is spontaneously broken in such a way that invariance under the electromagnetic subgroup $U(1)_{em}$ is preserved. The SM is chiral since right- and left-handed fermions transform according to different representations of the gauge group. Consequently fermion masses are forbidden in the symmetric theory. They are generated through spontaneous symmetry breaking from the Yukawa couplings. Diagonalization of the fermion mass introduces the quark mixing matrix in the quark

sector. This can give rise to CP-violation. Fermions appear in generations. The model does not fix their number, but from experiment we know that there are exactly three with light neutrinos [5].

The SM is a consistent quantum field theory. It is renormalizable, as was proven by 't HOOFT [17], and free of anomalies. Therefore it allows to calculate unique quantum corrections. Given a finite set of input parameters measurable quantities can be predicted order by order in perturbation theory.

The classical Lagrangian \mathcal{L}_C of the SM is composed of a Yang-Mills, a Higgs and a fermion part

$$\mathcal{L}_C = \mathcal{L}_{YM} + \mathcal{L}_H + \mathcal{L}_F. \tag{2.1}$$

Each of them is separately gauge invariant. They are specified as follows:

2.1. The Yang-Mills-part

The gauge fields are four vector fields transforming according to the adjoint representation of the gauge group $SU(2)_W \times U(1)_Y$. The isotriplet W_μ^a , $a=1, 2, 3$ is associated with the generators I_W^a of the weak isospin group $SU(2)_W$, the isosinglet B_μ with the weak hypercharge Y_W of the group $U(1)_Y$. The pure gauge field Lagrangian reads

$$\mathcal{L}_{YM} = -\frac{1}{4} (\hat{\partial}_\mu W_\nu^a - \partial_\nu W_\mu^a + g_2 \epsilon^{abc} W_\mu^b W_\nu^c)^2 - \frac{1}{4} (\hat{\partial}_\mu B_\nu - \hat{\partial}_\nu B_\mu)^2, \tag{2.2}$$

where ϵ^{abc} are the totally antisymmetric structure constants of $SU(2)$. Since the gauge group is non-simple there are two gauge coupling constants, the $SU(2)_W$ gauge coupling g_2 and the $U(1)_Y$ gauge coupling g_1 . The covariant derivative is given by

$$D_\mu = \hat{\partial}_\mu - i g_2 I_W^a W_\mu^a + i g_1 \frac{Y_W}{2} B_\mu. \tag{2.3}$$

The electric charge operator Q is composed of the weak isospin generator I_W^3 and the weak hypercharge according to the Gell-Mann Nishijima relation

$$Q = I_W^3 + \frac{Y_W}{2}. \tag{2.4}$$

2.2. The Higgs part

The minimal Higgs sector consists of a single complex scalar $SU(2)_W$ doublet field with hypercharge $Y_W = 1$

$$\Phi(x) = \begin{pmatrix} \phi^+(x) \\ \phi^0(x) \end{pmatrix}. \tag{2.5}$$

It is coupled to the gauge fields with the covariant derivative (2.3) and has a self coupling resulting in the Lagrangian

$$\mathcal{L}_H = (D_\mu \Phi)^\dagger (D^\mu \Phi) - V(\Phi). \tag{2.6}$$

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2.3. Fermio

The left-handed $SU(2)_W$ doublet:

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$$l_j^R = \omega$$

where $\omega_\pm = \frac{1 \pm \gamma_5}{2}$ generation index and down-type multiplets is reproduced by neutrinos. These masses, which ha

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The Higgs potential

$$V(\Phi) = \frac{\lambda}{4} (\Phi^\dagger \Phi)^2 - \mu^2 \Phi^\dagger \Phi \tag{2.7}$$

is constructed in such a way that it gives rise to spontaneous symmetry breaking. This means that the parameters λ and μ are chosen such that the potential $V(\Phi)$ takes its minimum for a nonvanishing Higgs field, i.e. the vacuum expectation value $\langle \Phi \rangle$ of the Higgs field is nonzero.

2.3. Fermionic Part

The left-handed fermions of each lepton (L) and quark (Q) generation are grouped into $SU(2)_W$ doublets (we suppress the colour index)

$$L_j^L = \omega_- L_j = \begin{pmatrix} \nu_j^L \\ l_j^L \end{pmatrix}, \quad Q_j^L = \omega_- Q_j = \begin{pmatrix} u_j^L \\ d_j^L \end{pmatrix}, \tag{2.8}$$

the right-handed fermions into singlets

$$l_j^R = \omega_+ l_j, \quad u_j^R = \omega_+ u_j, \quad d_j^R = \omega_+ d_j, \tag{2.9}$$

where $\omega_\pm = \frac{1 \pm \gamma_5}{2}$ is the projector on right- and left-handed fields, respectively. j is the generation index and ν, l, u and d stand for neutrinos, charged leptons, up-type quarks and down-type quarks, respectively. The weak hypercharge of the right- and left-handed multiplets is chosen such that the known electromagnetic charges of the fermions are reproduced by the Gell-Mann-Nishijima relation (2.4). There are no right-handed neutrinos. These could be easily added, but they would induce nonvanishing neutrino masses, which have not been observed experimentally so far.

The fermionic part of the Lagrangian reads

$$\begin{aligned} \mathcal{L}_F = & \sum_i (\bar{L}_i^L i \gamma^\mu D_\mu L_i^L + \bar{Q}_i^L i \gamma^\mu D_\mu Q_i^L) \\ & + \sum_i (\bar{l}_i^R i \gamma^\mu D_\mu l_i^R + \bar{u}_i^R i \gamma^\mu D_\mu u_i^R + \bar{d}_i^R i \gamma^\mu D_\mu d_i^R) \\ & - \sum_{ij} (\bar{L}_i^L G_{ij}^l l_j^R \Phi + \bar{Q}_i^L G_{ij}^u u_j^R \tilde{\Phi} + \bar{Q}_i^L G_{ij}^d d_j^R \Phi + \text{h.c.}). \end{aligned} \tag{2.10}$$

Note that in the covariant derivative D_μ acting on right-handed fermions the term involving g_2 is absent, since they are $SU(2)_W$ singlets. The primed fermion fields are by definition eigenstates of the electroweak gauge interaction, i.e. the covariant derivatives are diagonal in this basis with respect to the generation indices. G_{ij}^l, G_{ij}^u and G_{ij}^d are the Yukawa coupling matrices, $\tilde{\Phi} = (\phi^{0*}, -\phi^-)^T$ is the charge conjugated Higgs field and $\phi^- = (\phi^+)^*$. The $SU(2)_W \times U(1)_Y$ symmetry forbids explicit mass terms for the fermions. The masses of the fermions are generated through the Yukawa couplings via spontaneous symmetry breaking.

2.4. Physical fields and parameters

The theory is constructed such that the classical ground state of the scalar field satisfies

$$|\langle \Phi \rangle|^2 = \frac{2\mu^2}{\lambda} = \frac{v^2}{2} \neq 0. \tag{2.11}$$

In perturbation theory one has to expand around the ground state. Its phase is chosen such that the electromagnetic gauge invariance $U(1)_{em}$ is preserved and the Higgs field is written as

$$\Phi(x) = \begin{pmatrix} \phi^+(x) \\ \frac{1}{\sqrt{2}}(v + H(x) + i\chi(x)) \end{pmatrix}, \tag{2.12}$$

where the components ϕ^+ , H and χ have zero vacuum expectation values. ϕ^+ , ϕ^- and χ are unphysical degrees of freedom and can be eliminated by a suitable gauge transformation. The gauge in which they are absent is called unitary. The field H is the physical Higgs field with mass

$$M_H = \sqrt{2}\mu. \tag{2.13}$$

Inserting (2.12) into \mathcal{L}_C the vacuum expectation value v introduces couplings with mass dimension and mass terms for the gauge bosons and fermions.

The physical gauge boson and fermion fields are obtained by diagonalizing the corresponding mass matrices

$$\begin{aligned} W_\mu^\pm &= \frac{1}{\sqrt{2}}(W_\mu^1 \mp iW_\mu^2), \\ \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} &= \begin{pmatrix} c_W & s_W \\ -s_W & c_W \end{pmatrix} \begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix}, \\ f_i^L &= U_{ik}^{f,L} f_k^{\prime L}, \\ f_i^R &= U_{ik}^{f,R} f_k^{\prime R}, \end{aligned} \tag{2.14}$$

where

$$c_W = \cos \theta_W = \frac{g_2}{\sqrt{g_2^2 + g_1^2}}, \quad s_W = \sin \theta_W, \tag{2.15}$$

with the weak mixing angle θ_W and f stands for ν , l , u or d . The resulting masses are

$$\begin{aligned} M_W &= \frac{1}{2} g_2 v, \quad M_Z = \frac{1}{2} \sqrt{g_1^2 + g_2^2} v, \\ M_\nu &= 0, \quad m_{f,i} = U_{ik}^{f,L} G_{km}^f U_{mi}^{f,R\dagger} \frac{v}{\sqrt{2}}. \end{aligned} \tag{2.16}$$

The neutrinos ren the Yukawa coupl weak mixing angle

$$c_W = \frac{M_W}{M_Z}$$

Identifying the charge $e = \sqrt{4\pi\alpha}$

$$e = \frac{g_1 g_2}{\sqrt{g_1^2 + g_2^2}}$$

or

$$g_1 = \frac{e}{c_W},$$

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The neutrinos remain massless since the absence of the right-handed neutrinos forbids the Yukawa couplings which would generate their masses. With (2.16) we find for the weak mixing angle

$$c_W = \frac{M_W}{M_Z}. \tag{2.17}$$

Identifying the coupling of the photon field A_μ to the electron with the electrical charge $e = \sqrt{4\pi\alpha}$ yields

$$e = \frac{g_1 g_2}{\sqrt{g_1^2 + g_2^2}}, \tag{2.18}$$

or

$$g_1 = \frac{e}{c_W}, \quad g_2 = \frac{e}{s_W}. \tag{2.19}$$

The diagonalization of the fermion mass matrices introduces a matrix into the quark- W -boson couplings, the unitary quark mixing matrix

$$V_{ij} = U_{ik}^{u,L} U_{kj}^{d,L\dagger}. \tag{2.20}$$

There is no corresponding matrix in the lepton sector. Since there is no neutrino mass matrix, $U^{v,L}$ is completely arbitrary and can be chosen such that it cancels $U^{l,L}$ in the lepton- W -boson couplings. The same would also be true for the quark sector if all up-type or down-type quarks would be degenerate in masses. For degenerate masses one can choose $U^L = U^{R\dagger}$ arbitrary without destroying the diagonality of the corresponding mass matrix and thus eliminate V_{ij} .

The above relations (2.13), (2.16), (2.18), (2.20) allow to replace the original set of parameters

$$g_1, g_2, \lambda, \mu^2, G^l, G^u, G^d \tag{2.21}$$

by the parameters

$$e, M_W, M_Z, M_H, m_{f,i}, V_{ij} \tag{2.22}$$

which have a direct physical meaning. Thus we can express the Lagrangian (2.1) in terms of physical parameters and fields.

Inserting (2.12) into \mathcal{L}_C generates a term linear in the Higgs field H which we denote by $tH(x)$ with

$$t = v \left(\mu^2 - \frac{\lambda}{4} v^2 \right). \tag{2.23}$$

The tadpole t vanishes at lowest order due to the choice of v . We use t instead of v in the following. Choosing v as the correct vacuum expectation value of the Higgs field Φ is equivalent to the vanishing of t .

2.5. Quantization

Quantization of \mathcal{L}_C and higher order calculations require the specification of a gauge. We choose a renormalizable 't Hooft gauge with the following linear gauge fixings

$$\begin{aligned} F^\pm &= (\xi_1^W)^{-\frac{1}{2}} \partial^\mu W_\mu^\pm \mp i M_W (\xi_2^W)^{\frac{1}{2}} \phi^\pm, \\ F^Z &= (\xi_1^Z)^{-\frac{1}{2}} \partial^\mu Z_\mu - M_Z (\xi_2^Z)^{\frac{1}{2}} \chi, \\ F^\gamma &= (\xi_1^\gamma)^{-\frac{1}{2}} \partial^\mu A_\mu, \end{aligned} \tag{2.24}$$

leading to the following gauge fixing Lagrangian

$$\mathcal{L}_{\text{fix}} = -\frac{1}{2} [(F^\gamma)^2 + (F^Z)^2 + 2F^+ F^-]. \tag{2.25}$$

\mathcal{L}_{fix} involves the unphysical components of the gauge fields. In order to compensate their effects one introduces Faddeev Popov ghosts $u^\alpha(x), \bar{u}^\alpha(x)$ ($\alpha = \pm, \gamma, Z$) with the Lagrangian

$$\mathcal{L}_{\text{FP}} = \bar{u}^\alpha(x) \frac{\delta F^\alpha}{\delta \theta^\beta(x)} u^\beta(x). \tag{2.26}$$

$\frac{\delta F^\alpha}{\delta \theta^\beta(x)}$ is the variation of the gauge fixing operators F^α under infinitesimal gauge transformations characterized by $\theta^\beta(x)$.

The 't Hooft Feynman gauge $\xi^Z=1$ is particularly simple. At lowest order the poles of the ghost fields, unphysical Higgs fields and longitudinal gauge fields coincide with the poles of the corresponding transverse gauge fields. Furthermore no gauge-field-Higgs mixing occurs.

With \mathcal{L}_{fix} and \mathcal{L}_{FP} the complete renormalizable Lagrangian for the electroweak SM reads

$$\mathcal{L}_{\text{GSW}} = \mathcal{L}_C + \mathcal{L}_{\text{fix}} + \mathcal{L}_{\text{FP}}. \tag{2.27}$$

The corresponding Feynman rules are given in App. A.

3. Renormalization

The Lagrangian (2.1) of the minimal $SU(2)_W \times U(1)_Y$ model involves a certain number of free parameters (2.22) which have to be determined experimentally. These are chosen such that they have an intuitive physical meaning at tree level (physical masses, couplings), i.e. they are directly related to experimental quantities. This direct relation is destroyed through higher order corrections. Moreover the parameters of the original Lagrangian, the so-called bare parameters, differ from the corresponding physical quantities by UV-divergent contributions. However, in renormalizable theories these divergencies cancel in relations between physical quantities, thus allowing meaningful predictions. The renormalizability of nonabelian gauge theories with spontaneous symmetry breaking and thus of the SM was proven by 't Hooft [17].

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One possibility to evaluate predictions of a renormalizable model is the following:

- Calculate physical quantities in terms of the bare parameters.
- Use as many of the resulting relations as bare parameters exist to express these in terms of physical observables.
- Insert the resulting expressions into the remaining relations.

Thus one arrives at predictions for physical observables in terms of other physical quantities, which have to be determined from experiment. In these predictions all UV-divergencies cancel in any order of perturbation theory. The predictions obtained from different input parameters differ in finite orders of perturbation theory by higher order contributions. This treatment of renormalization has been pioneered by PASSARINO, VELTMAN and CONSOLI [18] and is the basis of the so-called "star" scheme of KENNEDY and LYNN [19].

We use the counterterm approach. Here the UV-divergent bare parameters are expressed by finite renormalized parameters and divergent renormalization constants (counterterms). In addition the bare fields may be replaced by renormalized fields. The counterterms are fixed through renormalization conditions. These can be chosen arbitrarily, but determine the relation between renormalized and physical parameters. Further evaluation proceeds like described above. The results depend in finite orders of perturbation theory not only on the choice of the input parameters but also on the choice of the renormalized parameters. Clearly the physical results are unambiguous up to the orders which have been taken into account completely. The renormalization procedure can be summarized as follows:

- Choose a set of independent parameters (e.g. (2.22) in the SM). *(physical observables)*
- Separate the bare parameters (and fields) into renormalized parameters (fields) and renormalization constants (see Sect. 3.1).
- Choose renormalization conditions to fix the counterterms (see Sect. 3.2).
- Express physical quantities in terms of the renormalized parameters.
- Choose input data in order to fix the values of the renormalized parameters.
- Evaluate predictions for physical quantities as functions of the input data.

The first three items in this list specify a renormalization scheme.

Putting the counterterms equal to zero, the renormalized parameters equal the bare parameters and we recover the first approach.

However, we can choose the counterterms such that the finite renormalized parameters are equal to physical parameters in all orders of perturbation theory. This is the so-called on-shell renormalization scheme. In the SM one uses the masses of the physical particles M_W, M_Z, M_H, m_f , the charge of the electron e and the quark mixing matrix V_{ij} as renormalized parameters. This scheme was proposed by ROSS and TAYLOR [20] and is widely used in the electroweak theory. The advantage of the on-shell scheme is that all parameters have a clear physical meaning and can be measured directly in suitable experiments²⁾. Furthermore the Thomson cross section from which e is obtained is exact to all orders of perturbation theory. However, not all of the particle masses are known experimentally with good accuracy. Therefore other schemes may sometimes be advantageous.

²⁾ This is not the case for the quark masses, due to the presence of the strong interaction. In practice these are replaced by suitable experimental input parameters (see Sect. 8.1).

Renormalization of the parameters is sufficient to obtain finite S-matrix elements, but it leaves Green functions divergent. This is due to the fact that radiative corrections change the normalization of the fields by an infinite amount. In order to get finite propagators and vertex functions the fields have to be renormalized, too. Furthermore radiative corrections provide nondiagonal corrections to the mass matrices so that the bare fields are no longer mass eigenstates. In order to rediagonalize the mass matrices one has to introduce matrix valued field renormalization constants. These allow to define the renormalized fields in such a way that they are the correct physical mass eigenstates in all orders of perturbation theory. If one does not renormalize the fields in this way, one needs a nontrivial wave function renormalization for the external particles. This is required in going from Green functions to S-matrix elements in order to obtain a properly normalized S-matrix.

The results for physical S-matrix elements are independent of the specific choice of field renormalization. There exist many different treatments in the literature [21, 22, 23, 24, 25]. Calculations without field renormalization were performed by [26].

3.1. Renormalization constants and counterterms

In the following we specify the on-shell renormalization scheme for the electroweak SM quantitatively. As independent parameters we choose the physical parameters specified in (2.22). The renormalized quantities and renormalization constants are defined as follows (we denote bare quantities by an index 0)

$$\begin{aligned}
 e_0 &= Z_e e = (1 + \delta Z_e) e, \\
 M_{W,0}^2 &= M_W^2 + \delta M_W^2, \\
 M_{Z,0}^2 &= M_Z^2 + \delta M_Z^2, \\
 M_{H,0}^2 &= M_H^2 + \delta M_H^2, \\
 m_{f,i,0} &= m_{f,i} + \delta m_{f,i}, \\
 V_{ij,0} &= (U_1 V U_2^\dagger)_{ij} = V_{ij} + \delta V_{ij}.
 \end{aligned} \tag{3.1}$$

U_1 and U_2 are unitary matrices since $V_{ij,0}$ and V_{ij} are both unitary.

Radiative corrections affect the Higgs potential in such a way that its minimum is shifted. In order to correct for this shift one introduces a counterterm to the vacuum expectation value of the Higgs field, which is determined such that the renormalized v is given by the actual minimum of the effective Higgs potential. Since we have replaced v by t (2.23) we must introduce a counterterm δt . This is fixed such that it cancels all tadpole diagrams, i.e. that the effective potential contains no term linear in the Higgs field H .

The counterterms defined above are sufficient to render all S-matrix elements finite. In order to have finite Green functions we must renormalize the fields, too. As explained above we need field renormalization matrices in order to be able to define renormalized fields which are mass eigenstates

$$\begin{aligned}
 W_0^\pm &= Z_W^{1/2} W^\pm = (1 + \frac{1}{2} \delta Z_W) W^\pm, \\
 \begin{pmatrix} Z_0 \\ A_0 \end{pmatrix} &= \begin{pmatrix} Z_{ZZ}^{1/2} & Z_{ZA}^{1/2} \\ Z_{AZ}^{1/2} & Z_{AA}^{1/2} \end{pmatrix} \begin{pmatrix} Z \\ A \end{pmatrix} = \begin{pmatrix} 1 + \frac{1}{2} \delta Z_{ZZ} & \frac{1}{2} \delta Z_{ZA} \\ \frac{1}{2} \delta Z_{AZ} & 1 + \frac{1}{2} \delta Z_{AA} \end{pmatrix} \begin{pmatrix} Z \\ A \end{pmatrix},
 \end{aligned}$$

$$\begin{aligned}
 H_0 &= Z_H^{1/2} H = (1 + \frac{1}{2} \delta Z_H) H, \\
 f_{i,0}^L &= Z_{ij}^{1/2, f, L} f_j^L = (\delta_{ij} + \frac{1}{2} \delta Z_{ij}^{f, L}) f_j^L, \\
 f_{i,0}^R &= Z_{ij}^{1/2, f, R} f_j^R = (\delta_{ij} + \frac{1}{2} \delta Z_{ij}^{f, R}) f_j^R.
 \end{aligned}
 \tag{3.2}$$

We do not discuss the renormalization constants of the unphysical ghost and Higgs fields. They do not affect Green functions of physical particles and are not relevant for the calculation of physical one-loop amplitudes. Furthermore the renormalization of the unphysical sector decouples from the one of the physical sector. It is governed by the Slavnov-Taylor identities. A discussion of this subject can be found e.g. in [24, 25].

In writing $Z = 1 + \delta Z$ for the multiplicative renormalization constants (matrices) we can split the bare Lagrangian \mathcal{L}_0 into the basic Lagrangian \mathcal{L} and the counterterm Lagrangian $\delta\mathcal{L}$

$$\mathcal{L}_0 = \mathcal{L} + \delta\mathcal{L}.
 \tag{3.3}$$

\mathcal{L} has the same form as \mathcal{L}_0 but depends on renormalized parameters and fields instead of unrenormalized ones. $\delta\mathcal{L}$ yields the counterterms. The corresponding Feynman rules are listed in App. A. They give rise to counterterm diagrams which have to be added to the loop graphs. Since we are only interested in one-loop corrections, we neglect terms of order $(\delta Z)^2$ everywhere.

3.2. Renormalization conditions

The renormalization constants introduced in the previous section are fixed by imposing renormalization conditions. These decompose into two sets. The conditions which define the renormalized parameters and the ones which define the renormalized fields. While the choice of the first affects physical predictions to finite orders of perturbation theory, the second are only relevant for Green functions and drop out when calculating S-matrix elements. Nevertheless their use is very convenient in the on-shell scheme. They not only allow to eliminate the explicit wave function renormalization of the external particles, but also simplify the explicit form of the renormalization conditions for the physical parameters considerably.

In the one-shell scheme all renormalization conditions are formulated for on mass shell external fields. The field renormalization constants, the mass renormalization constants and the renormalization constant of the quark mixing matrix are fixed using the one-particle irreducible two-point functions. For the charge renormalization we need one three-point function. For this we choose the $ee\gamma$ -vertex function. In the following renormalized quantities are denoted by the same symbols as the corresponding unrenormalized quantities, but with the superscript $\hat{}$.

As discussed above the first renormalization condition involves the tadpole T , the Higgs field one-point amputated renormalized Green function

$$\hat{T} = \text{---} \text{---} \text{---} \bigcirc \text{---} \text{---} \text{---},
 \tag{3.4}$$

(where the circle contains diagonal hatching lines)

and simply states

$$\hat{T} = T + \delta t = 0.
 \tag{3.5}$$

As a consequence of this condition no tadpoles need to be considered in actual calculations.

Next we need the renormalized one-particle irreducible two-point functions. These are defined as follows (we are using the 't Hooft-Feynman gauge)

$$\begin{aligned}
 & \text{Diagram: } W_\mu \text{ (wavy line, } k) \text{ --- } \text{circle} \text{ --- } W_\nu \text{ (wavy line)} = \hat{\Gamma}_{\mu\nu}^W(k) \\
 & = -i g_{\mu\nu} (k^2 - M_W^2) - i \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \hat{\Sigma}_T^W(k^2) - i \frac{k_\mu k_\nu}{k^2} \hat{\Sigma}_L^W(k^2),
 \end{aligned}$$

$$\begin{aligned}
 & \text{Diagram: } a, \mu \text{ (wavy line, } k) \text{ --- } \text{circle} \text{ --- } b, \nu \text{ (wavy line)} = \hat{\Gamma}_{\mu\nu}^{ab}(k) \\
 & = -i g_{\mu\nu} (k^2 - M_a^2) \delta_{ab} - i \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \hat{\Sigma}_T^{ab}(k^2) - i \frac{k_\mu k_\nu}{k^2} \hat{\Sigma}_L^{ab}(k^2),
 \end{aligned}$$

where

$$a, b = A, Z, \quad M_A^2 = 0,$$

$$\text{Diagram: } H \text{ (dashed line, } k) \text{ --- } \text{circle} \text{ --- } H \text{ (dashed line)} = \hat{\Gamma}^H(k) = i(k^2 - M_H^2) + i \hat{\Sigma}^H(k^2),$$

$$\text{Diagram: } f_j \text{ (solid line, } p) \text{ --- } \text{circle} \text{ --- } f_i \text{ (solid line)} = \hat{\Gamma}_{ij}^f(p)$$

$$= i \delta_{ij} (p - m_{f,i}) + i [p \omega_- \hat{\Sigma}_{ij}^{f,L}(p^2) + p \omega_- \hat{\Sigma}_{ij}^{f,R}(p^2) + (m_{f,i} \omega_- + m_{f,j} \omega_+) \hat{\Sigma}_{ij}^{f,S}(p^2)]. \quad (3.6)$$

The corresponding propagators are obtained as the inverse of these two-point functions. Note that we have to invert matrices for the neutral gauge bosons and for the fermions.

The renormalized mass parameters of the physical particles are fixed by the requirement that they are equal to the physical masses, i.e. to the real parts of the poles of the corresponding propagators which are equivalent to the zeros of the one-particle irreducible two-point functions. In case of mass matrices these conditions have to be fulfilled by the corresponding eigenvalues resulting in complicated expressions. These can be considerably simplified by requiring simultaneously the on-shell conditions for the field renormalization matrices. These state that the renormalized one-particle irreducible two-point functions are diagonal if the external lines are on their mass shell. This determines the nondiagonal elements of the field renormalization matrices. The diagonal elements are fixed such that the renormalized fields are properly normalized, i.e. that the residues of the renormalized propagators are equal to one. This choice of field renormalization implies that the renormalization conditions for the mass parameters (in all orders of perturbation theory) involve only the corresponding diagonal self energies. Thus we arrive at the following renormalization conditions for the two-point functions for on-shell external physical fields

$$\text{Re } \hat{\Gamma}_{\mu\nu}^W(k) \varepsilon^\nu$$

$$\text{Re } \hat{\Gamma}_{\mu\nu}^{ZZ}(k) \varepsilon^\nu$$

$$\hat{\Gamma}_{\mu\nu}^{AZ}(k) \varepsilon^\nu(k)$$

$$\lim_{k^2 \rightarrow M_W^2} \frac{1}{k^2 - M_W^2}$$

$$\lim_{k^2 \rightarrow M_Z^2} \frac{1}{k^2 - M_Z^2}$$

$$\text{Re } \hat{\Gamma}^H(k)|_{k^2 = M_H^2}$$

$$\text{Re } \hat{\Gamma}_{ij}^f(p) u_j$$

$$\lim_{p^2 \rightarrow m_{f,i}^2} \frac{p + m_{f,i}}{p^2 - m_{f,i}^2}$$

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$$\begin{aligned}
 & \widetilde{\text{Re}} \widehat{\Gamma}_{\mu\nu}^W(k) \varepsilon^\nu(k)|_{k^2=M_W^2} = 0, \\
 & \text{Re} \widehat{\Gamma}_{\mu\nu}^{ZZ}(k) \varepsilon^\nu(k)|_{k^2=M_Z^2} = 0, & \text{Re} \widehat{\Gamma}_{\mu\nu}^{AZ}(k) \varepsilon^\nu(k)|_{k^2=M_Z^2} = 0, \\
 & \widehat{\Gamma}_{\mu\nu}^{AZ}(k) \varepsilon^\nu(k)|_{k^2=0} = 0, & \widehat{\Gamma}_{\mu\nu}^{AA}(k) \varepsilon^\nu(k)|_{k^2=0} = 0, \\
 & \lim_{k^2 \rightarrow M_W^2} \frac{1}{k^2 - M_W^2} \widetilde{\text{Re}} \widehat{\Gamma}_{\mu\nu}^W(k) \varepsilon^\nu(k) = -i\varepsilon_\mu(k), \\
 & \lim_{k^2 \rightarrow M_Z^2} \frac{1}{k^2 - M_Z^2} \text{Re} \widehat{\Gamma}_{\mu\nu}^{ZZ}(k) \varepsilon^\nu(k) = -i\varepsilon_\mu(k), & \lim_{k^2 \rightarrow 0} \frac{1}{k^2} \text{Re} \widehat{\Gamma}_{\mu\nu}^{AA}(k) \varepsilon^\nu(k) = -i\varepsilon_\mu(k), \\
 & \text{Re} \widehat{\Gamma}^H(k)|_{k^2=M_H^2} = 0, & \lim_{k^2 \rightarrow M_H^2} \frac{1}{k^2 - M_H^2} \text{Re} \widehat{\Gamma}^H(k) = i, \\
 & \widetilde{\text{Re}} \widehat{\Gamma}_{ij}^f(p) u_j(p)|_{p^2=m_{f,i}^2} = 0, & \widetilde{\text{Re}} \bar{u}_i(p') \widehat{\Gamma}_{ij}^f(p')|_{p'^2=m_{f,i}^2} = 0, \\
 & \lim_{p^2 \rightarrow m_{f,i}^2} \frac{p + m_{f,i}}{p^2 - m_{f,i}^2} \widetilde{\text{Re}} \widehat{\Gamma}_{ii}^f(p) u_i(p) = i u_i(p), & \lim_{p'^2 \rightarrow m_{f,i}^2} \bar{u}_i(p') \widetilde{\text{Re}} \widehat{\Gamma}_{ii}^f(p') \frac{p' + m_{f,i}}{p'^2 - m_{f,i}^2} = i \bar{u}_i(p').
 \end{aligned} \tag{3.7}$$

$\varepsilon(k)$, $u(p)$ and $\bar{u}(p')$ are the polarization vectors and spinors of the external fields. $\widetilde{\text{Re}}$ takes the real part of the loop integrals appearing in the self energies but not of the quark mixing elements appearing there. Since we restrict ourselves to the one-loop order we apply it only to those quantities which depend on the quark mixing matrix at one loop. In higher orders Re must be replaced by $\widetilde{\text{Re}}$ everywhere. Re and $\widetilde{\text{Re}}$ are only relevant above thresholds and have no effect for the two-point functions of on-shell stable particles. If the quark mixing matrix is real $\widetilde{\text{Re}}$ can be replaced by Re . This holds in particular for a unit quark mixing matrix which is often used.

From the above equations we obtain the conditions for the self energy functions.

(3.6)

$$\begin{aligned}
 & \widetilde{\text{Re}} \widehat{\Sigma}_T^W(M_W^2) = 0, \\
 & \text{Re} \widehat{\Sigma}_T^{ZZ}(M_Z^2) = 0, & \text{Re} \widehat{\Sigma}_T^{AZ}(M_Z^2) = 0, \\
 & \widehat{\Sigma}_T^{AZ}(0) = 0, & \widehat{\Sigma}_T^{AA}(0) = 0, \text{ } ^3) \\
 & \widetilde{\text{Re}} \left. \frac{\partial \widehat{\Sigma}_T^W(k^2)}{\partial k^2} \right|_{k^2=M_W^2} = 0, \\
 & \text{Re} \left. \frac{\partial \widehat{\Sigma}_T^{ZZ}(k^2)}{\partial k^2} \right|_{k^2=M_Z^2} = 0, & \text{Re} \left. \frac{\partial \widehat{\Sigma}_T^{AA}(k^2)}{\partial k^2} \right|_{k^2=0} = 0, \\
 & \text{Re} \widehat{\Sigma}^H(M_H^2) = 0, & \text{Re} \left. \frac{\partial \widehat{\Sigma}^H(k^2)}{\partial k^2} \right|_{k^2=M_H^2} = 0,
 \end{aligned} \tag{3.8}$$

³⁾ This condition is automatically fulfilled due to a Ward identity.

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$$\begin{aligned}
m_{f,j} \tilde{\text{Re}} \hat{\Sigma}_{ij}^{f,L}(m_{f,j}^2) + m_{f,j} \tilde{\text{Re}} \hat{\Sigma}_{ij}^{f,S}(m_{f,j}^2) &= 0, \\
m_{f,j} \tilde{\text{Re}} \hat{\Sigma}_{ij}^{f,R}(m_{f,j}^2) + m_{f,i} \tilde{\text{Re}} \hat{\Sigma}_{ij}^{f,S}(m_{f,j}^2) &= 0, \\
\tilde{\text{Re}} \hat{\Sigma}_{ii}^{f,R}(m_{f,i}^2) + \tilde{\text{Re}} \hat{\Sigma}_{ii}^{f,L}(m_{f,i}^2) \\
+ 2m_{f,i}^2 \frac{\partial}{\partial p^2} (\tilde{\text{Re}} \hat{\Sigma}_{ii}^{f,R}(p^2) + \tilde{\text{Re}} \hat{\Sigma}_{ii}^{f,L}(p^2) + 2\tilde{\text{Re}} \hat{\Sigma}_{ii}^{f,S}(p^2))|_{p^2=m_{f,i}^2} &= 0. \tag{3.10}
\end{aligned}$$

Note that the (unphysical) longitudinal part of the gauge boson self energies drops out for on-shell external gauge bosons.

Our choice for the renormalization condition of the quark mixing matrix V_{ij} can be motivated as follows. To lowest order V_{ij} is given by (see eq. (2.20))

$$V_{0,ij} = U_{ik}^{u,L} U_{kj}^{d,L\dagger}, \tag{3.11}$$

where the matrices $U^{f,L}$ transform the weak interaction eigenstates f'_0 to the lowest order mass eigenstates f_0

$$U_{ij}^{f,L\dagger} f'_{j,0} = f_{i,0}. \tag{3.12}$$

In the on-shell renormalization scheme the higher order mass eigenstates are related to the bare mass eigenstates through the field renormalization constants of the fermions

$$f_i^L = Z_{ij}^{1/2, f,L} f_{j,0}^L. \tag{3.13}$$

We define the renormalized quark mixing matrix in analogy to the unrenormalized one through the rotation from the weak interaction eigenstates to the renormalized mass eigenstates. In the one-loop approximation the rotation contained in the fermion wave function renormalization $1 + \frac{1}{2} \delta Z^L$ is simply given by the anti-Hermitian part δZ^{AH} of δZ^L

$$\delta Z_{ij}^{f,AH} = \frac{1}{2} (\delta Z_{ij}^{f,L} - \delta Z_{ij}^{f,L\dagger}). \tag{3.14}$$

Thus we are lead to define the renormalized quark mixing matrix as

$$\begin{aligned}
V_{ij} &= (\delta_{ik} + \frac{1}{2} \delta Z_{ik}^{u,AH\dagger}) U_{km}^{u,L} U_{mn}^{d,L\dagger} (\delta_{nj} + \frac{1}{2} \delta Z_{nj}^{d,AH}) \\
&= (\delta_{ik} + \frac{1}{2} \delta Z_{ik}^{u,AH\dagger}) V_{0,kn} (\delta_{nj} + \frac{1}{2} \delta Z_{nj}^{d,AH}). \tag{3.15}
\end{aligned}$$

It has been shown that this condition correctly cancels all one-loop divergencies and that $V_{ij} = V_{0,ij}$ in the limit of degenerate up- or down-type quark masses [27].

Finally the electrical charge is defined as the full $ee\gamma$ -coupling for on-shell external particles in the Thomson limit. This means that all corrections to this vertex vanish on-shell and for zero momentum transfer⁴⁾

(3.10)

= $ie\bar{u}(p)\gamma_\mu u(p)$. (3.16)

The momenta p, p' flow in the direction of the fermion arrows. Due to our choice for the field renormalization the corrections in the external legs vanish and we obtain the condition

(3.11)

$$\bar{u}(p) \hat{\Gamma}_\mu^{ee\gamma}(p, p) u(p)|_{p^2=m_e^2} = ie\bar{u}(p) \gamma_\mu u(p), \quad (3.17)$$

for the (amputated) vertex function

(3.12)

(3.18)

3.3. Explicit form of renormalization constants

The renormalized quantities defined in Sect. 3.2 consist of the unrenormalized ones and the counterterms as specified by the Feynman rules in App. A. The renormalization conditions allow to express the counterterms by the unrenormalized self energies at special external momenta. This is evident for all renormalization constants apart from the one for the electrical charge. In this case, however, we can use a Ward identity to eliminate the vertex function.

From conditions (3.5), (3.8), (3.9) we obtain for the gauge boson and Higgs sector

(3.14)

$$\delta t = -T,$$

$$\delta M_W^2 = \tilde{R}e \Sigma_T^W(M_W^2), \quad \delta Z_W = -\tilde{R}e \left. \frac{\partial \Sigma_T^W(k^2)}{\partial k^2} \right|_{k^2=M_W^2},$$

$$\delta M_Z^2 = \tilde{R}e \Sigma_T^{ZZ}(M_Z^2), \quad \delta Z_{ZZ} = -\tilde{R}e \left. \frac{\partial \Sigma_T^{ZZ}(k^2)}{\partial k^2} \right|_{k^2=M_Z^2},$$

⁴⁾ Due to the wave function renormalization of the external particles the self energy corrections in the external legs contribute only with a factor 1/2 to the S-matrix elements.

$$\begin{aligned}
\delta Z_{AZ} &= -2 \operatorname{Re} \frac{\Sigma_T^{AZ}(M_Z^2)}{M_Z^2}, & \delta Z_{ZA} &= 2 \frac{\Sigma_T^{AZ}(0)}{M_Z^2}, \\
\delta Z_{AA} &= - \left. \frac{\partial \Sigma_T^{AA}(k^2)}{\partial k^2} \right|_{k^2=0}, \\
\delta M_H^2 &= \operatorname{Re} \Sigma^H(M_H^2), & \delta Z_H &= - \operatorname{Re} \left. \frac{\partial \Sigma^H(k^2)}{\partial k^2} \right|_{k^2=M_H^2}.
\end{aligned} \tag{3.19}$$

In the fermion sector (3.10) yields

$$\begin{aligned}
\delta m_{f,i} &= \frac{m_{f,i}}{2} \tilde{\operatorname{Re}}(\Sigma_{ii}^{f,L}(m_{f,i}^2) + \Sigma_{ii}^{f,R}(m_{f,i}^2) + 2 \Sigma_{ii}^{f,S}(m_{f,i}^2)), \\
\delta Z_{ij}^{f,L} &= \frac{2}{m_{f,i}^2 - m_{f,j}^2} \tilde{\operatorname{Re}} [m_{f,j}^2 \Sigma_{ij}^{f,L}(m_{f,j}^2) + m_{f,i} m_{f,j} \Sigma_{ij}^{f,R}(m_{f,j}^2) \\
&\quad + (m_{f,i}^2 + m_{f,j}^2) \Sigma_{ij}^{f,S}(m_{f,j}^2)], \quad i \neq j, \\
\delta Z_{ij}^{f,R} &= \frac{2}{m_{f,i}^2 - m_{f,j}^2} \tilde{\operatorname{Re}} [m_{f,j}^2 \Sigma_{ij}^{f,R}(m_{f,j}^2) + m_{f,i} m_{f,j} \Sigma_{ij}^{f,L}(m_{f,j}^2) \\
&\quad + 2 m_{f,i} m_{f,j} \Sigma_{ij}^{f,S}(m_{f,j}^2)], \quad i \neq j, \\
\delta Z_{ii}^{f,L} &= - \tilde{\operatorname{Re}} \Sigma_{ii}^{f,L}(m_{f,i}^2) - m_{f,i}^2 \frac{\hat{c}}{\partial p^2} \tilde{\operatorname{Re}} [\Sigma_{ii}^{f,L}(p^2) + \Sigma_{ii}^{f,R}(p^2) + 2 \Sigma_{ii}^{f,S}(p^2)]|_{p^2=m_{f,i}^2}, \\
\delta Z_{ii}^{f,R} &= - \tilde{\operatorname{Re}} \Sigma_{ii}^{f,R}(m_{f,i}^2) - m_{f,i}^2 \frac{\hat{c}}{\partial p^2} \tilde{\operatorname{Re}} [\Sigma_{ii}^{f,L}(p^2) + \Sigma_{ii}^{f,R}(p^2) + 2 \Sigma_{ii}^{f,S}(p^2)]|_{p^2=m_{f,i}^2}.
\end{aligned} \tag{3.20}$$

The use of $\tilde{\operatorname{Re}}$ ensures reality of the renormalized Lagrangian. Furthermore it yields

$$\delta Z_{ij}^\dagger = \delta Z_{ij}(m_i^2 \leftrightarrow m_j^2), \tag{3.21}$$

and in particular

$$\delta Z_{ii}^\dagger = \delta Z_{ii}. \tag{3.22}$$

In the lepton sector we have $V_{ij} = \delta_{ij}$. Consequently all lepton self energies are diagonal and the off-diagonal lepton wave function renormalization constants are zero. The same holds for the quark sector if one replaces the quark mixing matrix by a unit matrix as is usually done in calculations of radiative corrections for high energy processes.

The renormalization constant for the quark mixing matrix V_{ij} can be directly read off from (3.15)

$$\delta V_{ij} = \frac{1}{4} [(\delta Z_{ik}^{u,L} - \delta Z_{ik}^{u,L\dagger}) V_{kj} - V_{ik} (\delta Z_{kj}^{d,L} - \delta Z_{kj}^{d,L\dagger})]. \tag{3.23}$$

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Inserting the fermion field renormalization constants (3.20) yields

$$\begin{aligned}
 \delta V_{ij} = & \frac{1}{2} \widetilde{\text{Re}} \left\{ \frac{1}{m_{u,i}^2 - m_{u,k}^2} [m_{u,k}^2 \Sigma_{ik}^{u,L}(m_{u,k}^2) + m_{u,i}^2 \Sigma_{ik}^{u,L}(m_{u,i}^2) \right. \\
 & + m_{u,i} m_{u,k} (\Sigma_{ik}^{u,R}(m_{u,k}^2) + \Sigma_{ik}^{u,R}(m_{u,i}^2)) \\
 & + (m_{u,k}^2 + m_{u,i}^2) (\Sigma_{ik}^{u,S}(m_{u,k}^2) + \Sigma_{ik}^{u,S}(m_{u,i}^2))] V_{kj} \\
 & - V_{ik} \frac{1}{m_{d,k}^2 - m_{d,j}^2} [m_{d,j}^2 \Sigma_{kj}^{d,L}(m_{d,j}^2) + m_{d,k}^2 \Sigma_{kj}^{d,L}(m_{d,k}^2) \\
 & + m_{d,k} m_{d,j} (\Sigma_{kj}^{d,R}(m_{d,j}^2) + \Sigma_{kj}^{d,R}(m_{d,k}^2)) \\
 & \left. + (m_{d,k}^2 + m_{d,j}^2) (\Sigma_{kj}^{d,S}(m_{d,k}^2) + \Sigma_{kj}^{d,S}(m_{d,j}^2))] \right\}. \tag{3.24}
 \end{aligned}$$

It remains to fix the charge renormalization constant δZ_e . This is determined from the $ee\gamma$ -vertex. To be more general we investigate the $ff\gamma$ -vertex for arbitrary fermions f . The renormalized vertex function reads

$$\widehat{\Gamma}_{ij,\mu}^{\gamma ff}(p, p') = -ie \delta_{ij} Q_f \gamma_\mu + ie \widehat{\Lambda}_{ij,\mu}^{\gamma ff}(p, p'). \tag{3.25}$$

For on-shell external fermions it can be decomposed as ($k = p' - p$)

$$\widehat{\Lambda}_{ij,\mu}^{\gamma ff}(p, p') = \delta_{ij} \left(\gamma_\mu \widehat{\Lambda}_V^f(k^2) - \gamma_\mu \gamma_5 \widehat{\Lambda}_A^f(k^2) + \frac{(p+p')_\mu}{2m_f} \widehat{\Lambda}_S^f(k^2) + \frac{(p'-p)_\mu}{2m_f} \gamma_5 \widehat{\Lambda}_P^f(k^2) \right). \tag{3.26}$$

Expressing the renormalized quantities by the unrenormalized ones and the counterterms and inserting this in the analogue of the renormalization condition (3.17) for arbitrary fermions we find, using the Gordon identities,

$$\begin{aligned}
 0 = & \bar{u}(p) \widehat{\Lambda}_{ii,\mu}^{\gamma ff}(p, p) u(p) \\
 = & \bar{u}(p) \gamma_\mu u(p) [-Q_f (\delta Z_e + \delta Z_{ii}^{f,V}) + \frac{1}{2} \delta Z_{AA} + A_V^f(0) + A_S^f(0) + v_f \frac{1}{2} \delta Z_{ZA}] \\
 & - \bar{u}(p) \gamma_\mu \gamma_5 u(p) [-Q_f \delta Z_{ii}^{f,A} + A_A^f(0) + a_f \frac{1}{2} \delta Z_{ZA}], \tag{3.27}
 \end{aligned}$$

where

$$\delta Z_{ii}^{f,V} = \frac{1}{2} (\delta Z_{ii}^{f,L} + \delta Z_{ii}^{f,R}), \quad \delta Z_{ii}^{f,A} = \frac{1}{2} (\delta Z_{ii}^{f,L} - \delta Z_{ii}^{f,R}), \tag{3.28}$$

and v_f, a_f are the vector and axialvector couplings of the Z-boson to the fermion f , given explicitly in (A.15). This yields in fact two conditions, namely

$$0 = -Q_f \left(\delta Z_e + \delta Z_{ii}^{f,V} + \frac{1}{2} \delta Z_{AA} \right) + A_V^f(0) + A_S^f(0) + v_f \frac{1}{2} \delta Z_{ZA}, \tag{3.29}$$

$$0 = -Q_f \delta Z_{ii}^{f,A} + A_A^f(0) + a_f \frac{1}{2} \delta Z_{ZA}. \tag{3.30}$$

The first one (3.29) for $f=e$ fixes the charge renormalization constant. The second (3.30) is automatically fulfilled due to a Ward identity which can be derived from the gauge invariance of the theory. The same Ward identity moreover yields

$$A_V^f(0) + A_S^f(0) - Q_f \delta Z_{ii}^{f,V} + a_f \frac{1}{2} \delta Z_{ZA} = 0. \quad (3.31)$$

Inserting this in (3.29) we finally find (using $v_f - a_f = -Q_f s_W / c_W$)

$$\delta Z_e = -\frac{1}{2} \delta Z_{AA} - \frac{s_W}{c_W} \frac{1}{2} \delta Z_{ZA} = \frac{1}{2} \left. \frac{\partial \Sigma_T^{AA}(k^2)}{\partial k^2} \right|_{k^2=0} - \frac{s_W}{c_W} \frac{\Sigma_T^{AZ}(0)}{M_Z^2}. \quad (3.32)$$

This result is independent of the fermion species, reflecting electric charge universality. Clearly it does not depend on a specific choice of field renormalization. Consequently the analogue of (3.17) holds for arbitrary fermions f .

In the on-shell scheme the weak mixing angle is a derived quantity. Following SIRLIN [26] we define it as

$$\sin^2 \theta_W = s_W^2 = 1 - \frac{M_W^2}{M_Z^2}, \quad (3.33)$$

using the renormalized gauge boson masses. This definition is independent of a specific process and valid to all orders of perturbation theory.

Since the dependent parameters s_W and c_W frequently appear, it is useful to introduce the corresponding counterterms

$$c_{W,0} = c_W + \delta c_W, \quad s_{W,0} = s_W + \delta s_W. \quad (3.34)$$

Because of (3.33) these are directly related to the counterterms to the gauge boson masses. To one-loop order we obtain

$$\begin{aligned} \frac{\delta c_W}{c_W} &= \frac{1}{2} \left(\frac{\delta M_W^2}{M_W^2} - \frac{\delta M_Z^2}{M_Z^2} \right) = \frac{1}{2} \tilde{\text{Re}} \left(\frac{\Sigma_T^W(M_W^2)}{M_W^2} - \frac{\Sigma_T^{ZZ}(M_Z^2)}{M_Z^2} \right), \\ \frac{\delta s_W}{s_W} &= -\frac{c_W^2}{s_W^2} \frac{\delta c_W}{c_W} = -\frac{1}{2} \frac{c_W^2}{s_W^2} \tilde{\text{Re}} \left(\frac{\Sigma_T^W(M_W^2)}{M_W^2} - \frac{\Sigma_T^{ZZ}(M_Z^2)}{M_Z^2} \right). \end{aligned} \quad (3.35)$$

We have now determined all renormalization constants in terms of unrenormalized self energies. In the next sections we will describe the methods to calculate these self energies and more general diagrams at the one-loop level.

4. One-Loop Integrals

Perturbative calculations at one-loop order involve integrals over the loop momentum. In this chapter we discuss their classification and techniques for their calculation. The methods described here are to a large extent based on the work of PASSARINO and VELTMAN [18], 't HOOFT and VELTMAN [28], and MELROSE [29].

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4.1. Definitions

The one-loop integrals in D dimensions are classified according to the number N of propagator factors in the denominator and the number P of integration momenta in the numerator. For $P + D - 2N \geq 0$ these integrals are UV-divergent. The divergencies are regularized by calculating the integrals in general dimensions $D \neq 4$ (dimensional regularization). The UV-divergencies drop out in renormalized quantities. For renormalizable theories we have $P \leq N$ and thus a finite number of divergent integrals.

We define the general one-loop tensor integral (see Fig. 4.1) as

$$T_{\mu_1 \dots \mu_P}^N(p_1, \dots, p_{N-1}, m_0, \dots, m_{N-1}) = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \frac{q_{\mu_1} \dots q_{\mu_P}}{D_0 D_1 \dots D_{N-1}} \tag{4.1}$$

with the denominator factors

$$D_0 = q^2 - m_0^2 + i\epsilon, \quad D_i = (q + p_i)^2 - m_i^2 + i\epsilon, \quad i = 1, \dots, N-1, \tag{4.2}$$

originating from the propagators in the Feynman diagram. Furthermore we introduce

$$p_{i0} = p_i \quad \text{and} \quad p_{ij} = p_i - p_j. \tag{4.3}$$

Evidently the tensor integrals are invariant under arbitrary permutations of the propagators D_i , $i \neq 0$ and totally symmetric in the Lorentz indices μ_k . $i\epsilon$ is an infinitesimal imaginary part which is needed to regulate singularities of the integrand. Its specific choice ensures causality. After integration it determines the correct imaginary parts of the logarithms and dilogarithms. The parameter μ has mass dimension and serves to keep the dimension of the integrals fixed for varying D . Conventionally T^N is denoted by the N th character of the alphabet, i.e. $T^1 \equiv A$, $T^2 \equiv B, \dots$, and the scalar integrals carry an index 0.

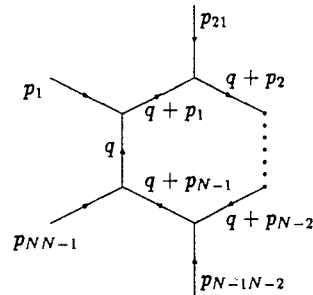


Figure 4.1: Conventions for the N-point integral

Lorentz covariance of the integrals allows to decompose the tensor integrals into tensors constructed from the external momenta p_i , and the metric tensor $g_{\mu\nu}$ with totally symmetric coefficient functions $T_{i_1 \dots i_P}^N$. We formally introduce an artificial momentum p_0 in order to write the terms containing $g_{\mu\nu}$ in a compact way

$$T_{\mu_1 \dots \mu_P}^N(p_1, \dots, p_{N-1}, m_0, \dots, m_{N-1}) = \sum_{i_1, \dots, i_P=0}^{N-1} T_{i_1 \dots i_P}^N p_{i_1 \mu_1} \dots p_{i_P \mu_P}. \tag{4.4}$$

From this formula the correct $g_{\mu\nu}$ terms are recovered by omitting all terms containing an odd number of p_0 's and replacing products of even numbers of p_0 's by the corresponding totally symmetric tensor constructed from the $g_{\mu\nu}$, e.g.

$$\begin{aligned} p_{0\mu_1} p_{0\mu_2} &\rightarrow g_{\mu_1\mu_2}, \\ p_{0\mu_1} p_{0\mu_2} p_{0\mu_3} p_{0\mu_4} &\rightarrow g_{\mu_1\mu_2} g_{\mu_3\mu_4} + g_{\mu_1\mu_3} g_{\mu_2\mu_4} + g_{\mu_1\mu_4} g_{\mu_2\mu_3}. \end{aligned} \quad (4.5)$$

The explicit Lorentz decompositions for the lowest order integrals read

$$\begin{aligned} B_\mu &= p_{1\mu} B_1, \\ B_{\mu\nu} &= g_{\mu\nu} B_{00} + p_{1\mu} p_{1\nu} B_{11}, \end{aligned} \quad (4.6)$$

$$\begin{aligned} C_\mu &= p_{1\mu} C_1 + p_{2\mu} C_2 = \sum_{i=1}^2 p_{i\mu} C_i, \\ C_{\mu\nu} &= g_{\mu\nu} C_{00} + p_{1\mu} p_{1\nu} C_{11} + p_{2\mu} p_{2\nu} C_{22} + (p_{1\mu} p_{2\nu} + p_{2\mu} p_{1\nu}) C_{12} \\ &= g_{\mu\nu} C_{00} + \sum_{i,j=1}^2 p_{i\mu} p_{j\nu} C_{ij}, \\ C_{\mu\nu\rho} &= (g_{\mu\nu} p_{1\rho} + g_{\nu\rho} p_{1\mu} + g_{\mu\rho} p_{1\nu}) C_{001} + (g_{\mu\nu} p_{2\rho} + g_{\nu\rho} p_{2\mu} + g_{\mu\rho} p_{2\nu}) C_{002} \\ &\quad + p_{1\mu} p_{1\nu} p_{1\rho} C_{111} + p_{2\mu} p_{2\nu} p_{2\rho} C_{222} \\ &\quad + (p_{1\mu} p_{1\nu} p_{2\rho} + p_{1\mu} p_{2\nu} p_{1\rho} + p_{2\mu} p_{1\nu} p_{1\rho}) C_{112} \\ &\quad + (p_{2\mu} p_{2\nu} p_{1\rho} + p_{2\mu} p_{1\nu} p_{2\rho} + p_{1\mu} p_{2\nu} p_{2\rho}) C_{122} \\ &= \sum_{i=1}^2 (g_{\mu\nu} p_{i\rho} + g_{\nu\rho} p_{i\mu} + g_{\mu\rho} p_{i\nu}) C_{00i} + \sum_{i,j,k=1}^2 p_{i\mu} p_{j\nu} p_{k\rho} C_{ijk}. \end{aligned} \quad (4.7)$$

$$\begin{aligned} D_\mu &= \sum_{i=1}^3 p_{i\mu} D_i, \\ D_{\mu\nu} &= g_{\mu\nu} D_{00} + \sum_{i,j=1}^3 p_{i\mu} p_{j\nu} D_{ij}, \\ D_{\mu\nu\rho} &= \sum_{i=1}^3 (g_{\mu\nu} p_{i\rho} + g_{\nu\rho} p_{i\mu} + g_{\mu\rho} p_{i\nu}) D_{00i} + \sum_{i,j,k=1}^3 p_{i\mu} p_{j\nu} p_{k\rho} D_{ijk}, \\ D_{\mu\nu\rho\sigma} &= (g_{\mu\nu} g_{\rho\sigma} + g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho}) D_{0000} \\ &\quad + \sum_{i,j=1}^3 (g_{\mu\nu} p_{i\rho} p_{j\sigma} + g_{\nu\rho} p_{i\mu} p_{j\sigma} + g_{\mu\rho} p_{i\nu} p_{j\sigma} \\ &\quad + g_{\mu\sigma} p_{i\nu} p_{j\rho} + g_{\nu\sigma} p_{i\mu} p_{j\rho} + g_{\rho\sigma} p_{i\mu} p_{j\nu}) D_{00ij} \\ &\quad + \sum_{i,j,k,l=1}^3 p_{i\mu} p_{j\nu} p_{k\rho} p_{l\sigma} D_{ijkl}. \end{aligned} \quad (4.8)$$

Since the four dimensional space is spanned by four Lorentz vectors the terms involving $g_{\mu\nu}$ should be omitted for $N \geq 5$ and at most four Lorentz vectors should be used

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in the decomposition (4.4). Consequently the Lorentz decomposition for a general tensor integral with $N \geq 5$ in four dimensions can be written as

$$T_{\mu_1 \dots \mu_P}^N(p_1, \dots, p_{N-1}, m_0, \dots, m_{N-1}) = \sum_{i_1, \dots, i_P=1}^4 T_{i_1 \dots i_P}^N p_{i_1 \mu_1} \dots p_{i_P \mu_P}, \tag{4.9}$$

where p_1, \dots, p_4 is any set of four linear independent Lorentz vectors out of p_1, \dots, p_{N-1} . The symmetry of the tensor integrals under exchange of the propagators yields relations between the scalar coefficient functions. Exchanging the arguments $(p_i, m_i) \leftrightarrow (p_j, m_j)$ together with the corresponding indices $i \leftrightarrow j$ leaves the scalar coefficient functions invariant

$$T_{\dots \overset{n}{i} \dots \overset{m}{j} \dots}^N(p_1, \dots, p_i, \dots, p_j, \dots, p_{N-1}, m_0, \dots, m_i, \dots, m_j, \dots, m_{N-1}) = T_{\dots \overset{m}{i} \dots \overset{n}{j} \dots}^N(p_1, \dots, p_j, \dots, p_i, \dots, p_{N-1}, m_0, \dots, m_j, \dots, m_i, \dots, m_{N-1}), \tag{4.10}$$

e.g.

$$\begin{aligned} C_1(p_1, p_2, m_0, m_1, m_2) &= C_2(p_2, p_1, m_0, m_2, m_1) \\ C_{00}(p_1, p_2, m_0, m_1, m_2) &= C_{00}(p_2, p_1, m_0, m_2, m_1), \\ C_{12}(p_1, p_2, m_0, m_1, m_2) &= C_{12}(p_2, p_1, m_0, m_2, m_1). \end{aligned} \tag{4.11}$$

All one-loop tensor integrals can be reduced to the scalar ones T_0^N . This is done in Sect. 4.2. General analytical results for the scalar integrals A_0, B_0, C_0 are D_0 are listed in Sect. 4.3. The scalar integrals for $N > 4$ can be expressed in terms of D_0 's in four dimensions. The relevant formulae are given in Sect. 4.4. They apply as well to the tensor integrals with $N \leq 4$ in the kinematical regions, where the usual tensor integral reduction breaks down, because the Gram determinants appearing there are zero. The UV-divergent parts of the one-loop integrals are explicitly given in Sect. 4.5.

4.2. Reduction of tensor integrals to scalar integrals

Using the Lorentz decomposition of the tensor integrals (4.4) the invariant functions $T_{i_1 \dots i_P}^N$ can be iteratively reduced to the scalar integrals T_0^N [18]. We derive the relevant formulae for the general tensor integral.

The product of the integration momentum q_μ with an external momentum can be expressed in terms of the denominators

$$q p_k = \frac{1}{2} [D_k - D_0 - f_k], \quad f_k = p_k^2 - m_k^2 + m_0^2. \tag{4.12}$$

Multiplying (4.1) with p_k and substituting (4.12) yields

$$\begin{aligned} R_{\mu_1 \dots \mu_{P-1}}^{N,k} &= T_{\mu_1 \dots \mu_P}^N p_k^\mu \\ &= \frac{1}{2} \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \left[\frac{q_{\mu_1} \dots q_{\mu_{P-1}}}{D_0 \dots D_{k-1} D_{k+1} \dots D_{N-1}} \right. \\ &\quad \left. - \frac{q_{\mu_1} \dots q_{\mu_{P-1}}}{D_1 \dots D_{N-1}} - f_k \frac{q_{\mu_1} \dots q_{\mu_{P-1}}}{D_0 \dots D_{N-1}} \right] \\ &= \frac{1}{2} [T_{\mu_1 \dots \mu_{P-1}}^{N-1}(k) - T_{\mu_1 \dots \mu_{P-1}}^{N-1}(0) - f_k T_{\mu_1 \dots \mu_{P-1}}^N], \end{aligned} \tag{4.13}$$

where the argument k of the tensor integrals in the last line indicates that the propagator D_k was cancelled. Note that $T_{\mu_1 \dots \mu_{P-1}}^{N-1}(0)$ has an external momentum in its first propagator. Therefore a shift of the integration momentum has to be performed in this integral in order to bring it to the form (4.1). All tensor integrals on the right-hand side of eq. (4.13) have one Lorentz index less than the original tensor integral. In two of them also one propagator is eliminated.

For $P \geq 2$ we obtain one more relation by contracting (4.1) with $g^{\mu\nu}$ and using

$$g^{\mu\nu} q_\mu q_\nu = q^2 = D_0 + m_0^2. \quad (4.14)$$

This gives

$$\begin{aligned} R_{\mu_1 \dots \mu_{P-2}}^{N,00} &= T_{\mu_1 \dots \mu_P}^N g^{\mu_{P-1}\mu_P} \\ &= \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \left[\frac{q_{\mu_1} \dots q_{\mu_{P-2}}}{D_1 \dots D_N} + m_0^2 \frac{q_{\mu_1} \dots q_{\mu_{P-2}}}{D_0 \dots D_N} \right] \\ &= [T_{\mu_1 \dots \mu_{P-2}}^{N-1}(0) + m_0^2 T_{\mu_1 \dots \mu_{P-2}}^N]. \end{aligned} \quad (4.15)$$

Inserting the Lorentz decomposition (4.4) for the tensor integrals T into (4.13) and (4.15) we obtain a set of linear equations for the corresponding coefficient functions. This set decomposes naturally into disjoint sets of $N-1$ equations for each tensor integral. If the inverse of the matrix

$$X_{N-1} = \begin{pmatrix} p_1^2 & p_1 p_2 & \dots & p_1 p_{N-1} \\ p_2 p_1 & p_2^2 & \dots & p_2 p_{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ p_{N-1} p_1 & p_{N-1} p_2 & \dots & p_{N-1}^2 \end{pmatrix} \quad (4.16)$$

exists, these can be solved for the invariant functions $T_{i_1 \dots i_p}^N$ yielding them in terms of invariant functions of tensor integrals with fewer indices (see eqs. (4.18) and (4.19) below). In this way all tensor integrals are expressed iteratively in terms of scalar integrals T_0^L with $L \leq N$.

If the matrix X_{N-1} becomes singular, the reduction algorithm breaks down. If this is due to the linear dependence of the momenta we can leave out the linear dependent vectors of the set p_1, \dots, p_{N-1} in the Lorentz decomposition resulting in a smaller matrix X_M . If X_M is nonsingular the reduction algorithm works again. This happens usually at the edge of phase space where some of the momenta p_i become collinear.

If the determinant of X_{N-1} , the Gram determinant, is zero but the momenta are not linear dependent⁵⁾ one has to use a different reduction algorithm [29, 30]. This will be discussed in Sect. 4.4.

Here we give the results for the reduction of arbitrary N -point integrals depending on $M \leq N-1$ linear independent Lorentz vectors in D dimensions for nonsingular X_M . Inserting the Lorentz decomposition of T_N and $R^{N,k}$ as well as $R^{N,00}$

$$\begin{aligned} R_{\mu_1 \dots \mu_{P-1}}^{N,k} &= T_{\mu_1 \dots \mu_P}^N p_k^{\mu_P} = \sum_{i_1, \dots, i_{P-1}=0}^M R_{i_1 \dots i_{P-1}}^{N,k} p_{i_1 \mu_1} \dots p_{i_{P-1} \mu_{P-1}}, \\ R_{\mu_1 \dots \mu_{P-2}}^{N,00} &= T_{\mu_1 \dots \mu_P}^N g^{\mu_{P-1}\mu_P} = \sum_{i_1, \dots, i_{P-2}=0}^M R_{i_1 \dots i_{P-2}}^{N,00} p_{i_1 \mu_1} \dots p_{i_{P-2} \mu_{P-2}}, \end{aligned} \quad (4.17)$$

⁵⁾ This can happen, because of the indefinite metric of space time.

into the first lines of (4.13) and (4.15) these equations can be solved for the $T_{i_1 \dots i_p}^N$:

$$T_{00i_1 \dots i_{P-2}}^N = \frac{1}{D + P - 2 - M} \left[R_{i_1 \dots i_{P-2}}^{N,00} - \sum_{k=1}^M R_{ki_1 \dots i_{P-2}}^{N,k} \right],$$

$$T_{ki_1 \dots i_{P-1}}^N = (X_M^{-1})_{kk'} \left[R_{i_1 \dots i_{P-1}}^{N,k'} - \sum_{r=1}^{P-1} \delta_{ir}^{k'} T_{00i_1 \dots i_{r-1} i_{r+1} \dots i_{P-1}}^N \right]. \quad (4.18)$$

Note that the numerator of the prefactor in the first equation is always positive in the relevant cases $P \geq 2$ and $D > M$. Using the third lines of (4.13) and (4.15) the R 's can be expressed in terms of $T_{i_1 \dots i_{P-1}}^N$, $T_{i_1 \dots i_{P-2}}^N$ and $T_{i_1 \dots i_q}^{N-1}$, with $q < P$ as follows

$$R_{i_1 \dots i_q \underbrace{M \dots M}_{P-2-q}}^{N,00} = m_0^2 T_{i_1 \dots i_q \underbrace{M \dots M}_{P-2-q}}^N$$

$$+ (-1)^{P-q} \left[\tilde{T}_{i_1 \dots i_q}^{N-1}(0) + \binom{P-2-q}{1} \sum_{k_1=1}^{M-1} \tilde{T}_{i_1 \dots i_q k_1}^{N-1}(0) \right.$$

$$+ \binom{P-2-q}{2} \sum_{k_1, k_2=1}^{M-1} \tilde{T}_{i_1 \dots i_q k_1 k_2}^{N-1}(0) + \dots$$

$$\left. + \binom{P-2-q}{P-2-q} \sum_{k_1, \dots, k_{P-2-q}=1}^{M-1} \tilde{T}_{i_1 \dots i_q k_1 \dots k_{P-2-q}}^{N-1}(0) \right], \quad (4.16)$$

$$R_{i_1 \dots i_q \underbrace{M \dots M}_{P-1-q}}^{N,k} = \frac{1}{2} \left\{ T_{i_1 \dots i_q \underbrace{M \dots M}_{P-1-q}}^{N-1}(k) \theta(k | i_1, \dots, i_q, \underbrace{M, \dots, M}_{P-1-q}) - f_k T_{i_1 \dots i_q \underbrace{M \dots M}_{P-1-q}}^N \right.$$

$$- (-1)^{P-1-q} \left[\tilde{T}_{i_1 \dots i_q}^{N-1}(0) + \binom{P-1-q}{1} \sum_{k_1=1}^{M-1} \tilde{T}_{i_1 \dots i_q k_1}^{N-1}(0) \right.$$

$$+ \binom{P-1-q}{2} \sum_{k_1, k_2=1}^{M-1} \tilde{T}_{i_1 \dots i_q k_1 k_2}^{N-1}(0) + \dots$$

$$\left. \left. + \binom{P-1-q}{P-1-q} \sum_{k_1, \dots, k_{P-1-q}=1}^{M-1} \tilde{T}_{i_1 \dots i_q k_1 \dots k_{P-1-q}}^{N-1}(0) \right] \right\} \quad (4.19)$$

where $i_1, \dots, i_q \neq M$ and

$$\theta(k | i_1, \dots, i_{P-1}) = \begin{cases} 1 & i_r \neq k, \quad r=1, \dots, P-1, \\ 0 & \text{else.} \end{cases} \quad (4.20)$$

The indices \tilde{i} refer to the i -th momentum of the corresponding N -point function T^N but to the $(i-1)$ -th momentum of the $N-1$ -point function $T^{N-1}(k)$ if $i > k$. Again the arguments of the T 's indicate the cancelled propagators. The tilde in $\tilde{T}(0)$ means that a shift of the integration variable $q \rightarrow q - p_M$ has been performed in order to obtain the standard form of these integrals. This shift generates the terms in the square brackets

(4.17)

of (4.19). It is also the reason for the unsymmetric appearance of the index M in the above equations. A different shift would result in similar results. An explicit example illustrating the use of these reduction formulae is given in App. C.

The recursion formulae above determine the coefficients $T_{i_1 \dots i_P}^N$ regardless of their symmetries. Consequently coefficients whose indices are not all equal are obtained in different ways. This allows for checks on the analytical results as well as on numerical stability.

If the number M of linear independent momenta equals the dimension D of space-time then the terms containing $g_{\mu\nu}$ in the Lorentz decomposition have to be omitted, since $g_{\mu\nu}$ can be built up from the D momenta. In this case the coefficients $T_{i_1 \dots i_P}^N$ are obtained from the second equations in (4.18) and (4.19) with $T_{00i_1 \dots i_{P-2}}^N = 0$.

4.3. Scalar one-loop integrals for $N \leq 4$

With the methods described in the last section all one-loop integrals can be reduced to the scalar ones T_0^N provided the matrices X_M are nonsingular. General analytical results for A_0 , B_0 , C_0 and D_0 were derived in [28]. Algorithms for the numerical calculation of the scalar one-loop integrals based on these results have been presented in [31]. Here we give a new formula [32] for D_0 involving only 16 dilogarithms compared to 24 of the solution of [28]. For completeness we first list the results for A_0 , B_0 and C_0 .

4.3.1. Scalar one-point function

The scalar one-point function reads

$$A_0(m) = -m^2 \left(\frac{m^2}{4\pi\mu^2} \right)^{\frac{D-4}{2}} \Gamma\left(1 - \frac{D}{2}\right) = m^2 \left(\Delta - \log \frac{m^2}{\mu^2} + 1 \right) + O(D-4), \quad (4.21)$$

with the UV-divergence contained in

$$\Delta = \frac{2}{4-D} - \gamma_E + \log 4\pi \quad (4.22)$$

and γ_E is Euler's constant. The terms of order $O(D-4)$ are only relevant for two- or higher-loop calculations.

4.3.2. Scalar two-point function

The two-point function is given by

$$\begin{aligned} B_0(p_{10}, m_0, m_1) &= \Delta - \int_0^1 dx \log \frac{[p_{10}^2 x^2 - x(p_{10}^2 - m_0^2 + m_1^2) + m_1^2 - i\varepsilon]}{\mu^2} + O(D-4) \\ &= \Delta + 2 - \log \frac{m_0 m_1}{\mu^2} + \frac{m_0^2 - m_1^2}{p_{10}^2} \log \frac{m_1}{m_0} - \frac{m_0 m_1}{p_{10}^2} \left(\frac{1}{r} - r \right) \log r \\ &\quad + O(D-4), \end{aligned} \quad (4.23)$$

For

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This is easily $\frac{\partial}{\partial p^2}$

4.3.3. Sc:

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physical ma $C_0(p_{10},$ $-\int_0^1 d.$ $+ (m_1^2$ $= \frac{1}{\alpha} \sum_{i=0}^2$ $+ \eta \left($ $- [\eta^1$ with $(i, j, k$ y_i x y α α

where r and $\frac{1}{r}$ are determined from

$$x^2 + \frac{m_0^2 + m_1^2 - p_{10}^2 - i\epsilon}{m_0 m_1} x + 1 = (x+r) \left(x + \frac{1}{r}\right). \tag{4.24}$$

The variable r never crosses the negative real axis even for complex physical masses (m^2 has a negative imaginary part!). For $r < 0$ the $i\epsilon$ prescription yields $\text{Im } r = \epsilon \text{sgn}(r - \frac{1}{r})$. Consequently the result (4.23) is valid for arbitrary physical parameters.

For the field renormalization constants we need the derivative of B_0 with respect to p_{10}^2 . This is easily obtained by differentiating the above result

$$\begin{aligned} \frac{\partial}{\partial p_{10}^2} B_0(p_{10}, m_0, m_1) &= -\frac{m_0^2 - m_1^2}{p_{10}^4} \log \frac{m_1}{m_0} + \frac{m_0 m_1}{p_{10}^4} \left(\frac{1}{r} - r\right) \log r \\ &\quad - \frac{1}{p_{10}^2} \left(1 + \frac{r^2 + 1}{r^2 - 1} \log r\right) + O(D-4). \end{aligned} \tag{4.25}$$

4.3.3. Scalar three-point function

The general result for the scalar three-point function valid for all real momenta and physical masses was calculated by [28]. It can be brought into the symmetric form

$$\begin{aligned} C_0(p_{10}, p_{20}, m_0, m_1, m_2) &= \\ &= -\int_0^1 dx \int_0^x dy [p_{21}^2 x^2 + p_{10}^2 y^2 + (p_{20}^2 - p_{10}^2 - p_{21}^2)xy \\ &\quad + (m_1^2 - m_2^2 - p_{21}^2)x + (m_0^2 + m_1^2 + p_{21}^2 - p_{20}^2)y + m_2^2 - i\epsilon]^{-1} \\ &= \frac{1}{\alpha} \sum_{i=0}^2 \left\{ \sum_{\sigma=\pm} \left[\text{Li}_2\left(\frac{y_{0i}-1}{y_{i\sigma}}\right) - \text{Li}_2\left(\frac{y_{0i}}{y_{i\sigma}}\right) \right. \right. \\ &\quad \left. \left. + \eta\left(1-x_{i\sigma}, \frac{1}{y_{i\sigma}}\right) \log \frac{y_{0i}-1}{y_{i\sigma}} - \eta\left(-x_{i\sigma}, \frac{1}{y_{i\sigma}}\right) \log \frac{y_{0i}}{y_{i\sigma}} \right] \right. \\ &\quad \left. - [\eta(-x_{i+}, -x_{i-}) - \eta(y_{i+}, y_{i-}) - 2\pi i \theta(-p_{jk}^2) \theta(-\text{Im}(y_{i+} y_{i-}))] \log \frac{1-y_{i0}}{-y_{i0}} \right\}, \end{aligned} \tag{4.26}$$

with $(i, j, k = 0, 1, 2$ and cyclic)

$$\begin{aligned} y_{0i} &= \frac{1}{2\alpha p_{jk}^2} [p_{jk}^2(p_{jk}^2 - p_{ki}^2 - p_{ij}^2 + 2m_i^2 - m_j^2 - m_k^2) \\ &\quad - (p_{ki}^2 - p_{ij}^2)(m_j^2 - m_k^2) + \alpha(p_{jk}^2 - m_j^2 + m_k^2)], \\ x_{i\pm} &= \frac{1}{2p_{jk}^2} [p_{jk}^2 - m_j^2 + m_k^2 \pm \alpha_i], \\ y_{i\pm} &= y_{0i} - x_{i\pm}, \\ \alpha &= \kappa(p_{10}^2, p_{21}^2, p_{20}^2), \\ \alpha_i &= \kappa(p_{jk}^2, m_j^2, m_k^2) (1 + i\epsilon p_{jk}^2), \end{aligned} \tag{4.27}$$

and κ is the Källén function

$$\kappa(x, y, z) = \sqrt{x^2 + y^2 + z^2 - 2(xy + yz + zx)}. \quad (4.28)$$

The dilogarithm or Spence function $\text{Li}_2(x)$ is defined as

$$\text{Li}_2(x) = -\int_0^1 \frac{dt}{t} \log(1 - xt), \quad |\arg(1 - x)| < \pi. \quad (4.29)$$

The η -function compensates for cut crossings on the Riemann-sheet of the logarithms and dilogarithms. For a, b on the first Riemann sheet it is defined by

$$\log(ab) = \log(a) + \log(b) + \eta(a, b). \quad (4.30)$$

All η -functions in (4.26) vanish if α and all the masses m_i are real. Note that α is real in particular for all on-shell decay and scattering processes.

4.3.4. Scalar four-point function

The scalar four-point function $D_0(p_{10}, p_{20}, p_{30}, m_0, m_1, m_2, m_3)$ can be expressed in terms of 16 dilogarithms [32].

Before we give the result we first introduce some useful variables and functions. We define

$$k_{ij} = \frac{m_i^2 + m_j^2 - p_{ij}^2}{m_i m_j}, \quad i, j = 0, 1, 2, 3, \quad (4.31)$$

and r_{ij} and \tilde{r}_{ij} by

$$x^2 + k_{ij}x + 1 = (x + r_{ij})(x + 1/r_{ij}), \quad (4.32)$$

and

$$x^2 + (k_{ij} - i\varepsilon)x + 1 = (x + \tilde{r}_{ij})(x + 1/\tilde{r}_{ij}). \quad (4.33)$$

Note that for real k_{ij} the r_{ij} 's lie either on the real axis or on the complex unit circle. Furthermore

$$P(y_0, y_1, y_2, y_3) = \sum_{0 \leq i < j \leq 3} k_{ij} y_i y_j + \sum_{j=0}^3 y_j^2, \quad (4.34)$$

$$Q(y_0, y_1, 0, y_3) = (1/r_{02} - r_{02})y_0 + (k_{12} - r_{02}k_{01})y_1 + (k_{23} - r_{02}k_{03})y_3, \quad (4.35)$$

$$\bar{Q}(y_0, 0, y_2, y_3) = (1/r_{13} - r_{13})y_3 + (k_{12} - r_{13}k_{23})y_2 + (k_{01} - r_{13}k_{03})y_0, \quad (4.36)$$

and $x_{1,2}$ is defined by

$$\begin{aligned}
 (4.28) \quad & \frac{r_{02}r_{13}}{x} \left\{ \left[P\left(1, \frac{x}{r_{13}}, 0, 0\right) - i\varepsilon \right] \left[P\left(0, 0, \frac{1}{r_{02}}, x\right) - i\varepsilon \right] \right. \\
 & \quad \left. - \left[P\left(0, \frac{x}{r_{13}}, \frac{1}{r_{02}}, 0\right) - i\varepsilon \right] \left[P(1, 0, 0, x) - i\varepsilon \right] \right\} \\
 (4.29) \quad & = ax^2 + bx + c + i\varepsilon d = a(x-x_1)(x-x_2), \tag{4.37}
 \end{aligned}$$

where

$$\begin{aligned}
 (4.30) \quad & a = k_{23}/r_{13} + r_{02}k_{01} - k_{03}r_{02}/r_{13} - k_{12}, \\
 & b = (r_{13} - 1/r_{13})(r_{02} - 1/r_{02}) + k_{01}k_{23} - k_{03}k_{12}, \\
 & c = k_{01}/r_{02} + r_{13}k_{23} - k_{03}r_{13}/r_{02} - k_{12}, \\
 & d = k_{12} - r_{02}k_{01} - r_{13}k_{23} + r_{02}r_{13}k_{03}. \tag{4.38}
 \end{aligned}$$

In addition we introduce

$$\gamma_{kl} = \text{sgn Re}[a(x_k - x_l)], \quad k, l = 1, 2, \tag{4.39}$$

and

$$\begin{aligned}
 (4.31) \quad & x_{k0} = x_k, \quad s_0 = \tilde{r}_{03}, \\
 & x_{k1} = x_k/r_{13}, \quad s_1 = \tilde{r}_{01}, \\
 (4.32) \quad & x_{k2} = x_k r_{02}/r_{13}, \quad s_2 = \tilde{r}_{12}, \\
 & x_{k3} = x_k r_{02}, \quad s_3 = \tilde{r}_{23}. \tag{4.40}
 \end{aligned}$$

as well as

$$(4.33) \quad x_{kj}^{(0)} = \lim_{\varepsilon \rightarrow 0} x_{kj} \quad \text{as} \quad r_{ij} = \lim_{\varepsilon \rightarrow 0} \tilde{r}_{ij}. \tag{4.41}$$

Finally we need

$$\begin{aligned}
 (4.34) \quad & \\
 (4.35) \quad & \tilde{\eta}(a, \tilde{b}) = \begin{cases} \eta(a, b) & \text{for } b \text{ not real,} \\ 2\pi i [\theta(-\text{Im } a) \theta(-\text{Im } \tilde{b}) - \theta(\text{Im } a) \theta(\text{Im } \tilde{b})] & \text{for } b < 0, \\ 0 & \text{for } b > 0, \end{cases} \tag{4.42}
 \end{aligned}$$

$$(4.36) \quad \text{with } b = \lim_{\varepsilon \rightarrow 0} \tilde{b}.$$

Then the result for real r_{02} can be written as

$$\begin{aligned}
 D_0(p_{10}, p_{20}, p_{30}, m_0, m_1, m_2, m_3) &= \frac{1}{m_1 m_2 m_3 m_4 a(x_1 - x_2)} \\
 &\times \left\{ \sum_{j=0}^3 \sum_{k=1}^2 (-1)^{j+k} \left[\text{Li}_2(1 + s_j x_{kj}) + \eta(-x_{kj}, s_j) \log(1 + s_j x_{kj}) \right. \right. \\
 &+ \left. \text{Li}_2\left(1 + \frac{x_{kj}}{s_j}\right) + \eta\left(-x_{kj}, \frac{1}{s_j}\right) \log\left(1 + \frac{x_{kj}}{s_j}\right) \right] \\
 &+ \sum_{k=1}^2 (-1)^{k+1} \left[\tilde{\eta}(-x_k, \tilde{r}_{02}) \left[\log(r_{02} x_k) + \log\left(Q\left(\frac{1}{x_k^{(0)}}, 0, 0, 1\right) - i\varepsilon\right) \right. \right. \\
 &+ \left. \left. \log\left(\frac{\bar{Q}(0, 0, 1, r_{02} x_k^{(0)})}{d} + i\varepsilon \gamma_{k,3-k} \text{sgn}(r_{02} \text{Im} \tilde{r}_{13})\right) \right] \right. \\
 &+ \left. \tilde{\eta}\left(-x_k, \frac{1}{\tilde{r}_{13}}\right) \left[\log\left(\frac{x_k}{r_{13}}\right) + \log\left(Q\left(\frac{r_{13}}{x_k^{(0)}}, 1, 0, 0\right) - i\varepsilon\right) \right. \right. \\
 &+ \left. \left. \log\left(\frac{\bar{Q}(1, 0, 0, x_k^{(0)})}{d} + i\varepsilon \gamma_{k,3-k} \text{sgn}(\text{Im} \tilde{r}_{13})\right) \right] \right. \\
 &- \left. \left[\tilde{\eta}\left(-x_k, \frac{\tilde{r}_{02}}{\tilde{r}_{13}}\right) + \eta\left(\tilde{r}_{02}, \frac{1}{\tilde{r}_{13}}\right) \right] \left[\log\left(\frac{r_{02} x_k}{r_{13}}\right) + \log\left(Q\left(\frac{r_{13}}{x_k^{(0)}}, 1, 0, 0\right) - i\varepsilon\right) \right. \right. \\
 &+ \left. \left. \log\left(\frac{\bar{Q}(0, 0, 1, r_{02} x_k^{(0)})}{d} + i\varepsilon \gamma_{k,3-k} \text{sgn}(r_{02} \text{Im} \tilde{r}_{13})\right) \right] \right. \\
 &+ \left. \left. \eta\left(\tilde{r}_{02}, \frac{1}{\tilde{r}_{13}}\right) \tilde{\eta}\left(-x_k, -\frac{\tilde{r}_{02}}{\tilde{r}_{13}}\right) \right] \right\}. \tag{4.43}
 \end{aligned}$$

In the case that $|r_{ij}| = 1$ for all r_{ij} , the result reads:

$$\begin{aligned}
 D_0(p_{10}, p_{20}, p_{30}, m_0, m_1, m_2, m_3) &= \frac{1}{m_1 m_2 m_3 m_4 a(x_1 - x_2)} \\
 &\times \left\{ \sum_{j=0}^3 \sum_{k=1}^2 (-1)^{j+k} \left[\text{Li}_2(1 + s_j x_{kj}) + \eta(-x_{kj}, s_j) \log(1 + s_j x_{kj}) \right. \right. \\
 &+ \left. \text{Li}_2\left(1 + \frac{x_{kj}}{s_j}\right) + \eta\left(-x_{kj}, \frac{1}{s_j}\right) \log\left(1 + \frac{x_{kj}}{s_j}\right) \right] \\
 &+ \sum_{k=1}^2 (-1)^{k+1} \left[\eta\left(-x_k, \frac{1}{r_{13}}\right) \left[\log\left(\frac{r_{13}}{x_k^{(0)}} P\left(1, \frac{x_k^{(0)}}{r_{13}}, 0, 0\right) - \frac{x_k^{(0)}}{r_{13}} \varepsilon b \gamma_{k,3-k}\right) + \log\left(\frac{x_k^{(0)}}{r_{13}}\right) \right] \right. \\
 &\left. \left. \eta\left(-x_k, \frac{1}{r_{13}}\right) \left[\log\left(\frac{r_{13}}{x_k^{(0)}} P\left(1, \frac{x_k^{(0)}}{r_{13}}, 0, 0\right) - \frac{x_k^{(0)}}{r_{13}} \varepsilon b \gamma_{k,3-k}\right) + \log\left(\frac{x_k^{(0)}}{r_{13}}\right) \right] \right] \right\}
 \end{aligned}$$

$$\begin{aligned}
 & + \eta(-x_k, r_{02}) \left[\log \left(\frac{1}{r_{02} x_k^{(0)}} P(0, 0, 1, r_{02} x_k^{(0)}) - r_{02} x_k^{(0)} \varepsilon b \gamma_{k, 3-k} \right) + \log(r_{02} x_k^{(0)}) \right] \\
 & - \left[\eta \left(-x_k, \frac{r_{02}}{r_{13}} \right) + \eta \left(r_{02}, \frac{1}{r_{13}} \right) \right] \left[\log \left(\frac{r_{13}}{r_{02} x_k^{(0)}} P \left(0, 1, \frac{r_{02} x_k^{(0)}}{r_{13}}, 0 \right) - \frac{r_{02} x_k^{(0)}}{r_{13}} \varepsilon b \gamma_{k, 3-k} \right) \right. \\
 & \left. + \log \left(\frac{r_{02} x_k^{(0)}}{r_{13}} \right) \right] + (1 - \gamma_{k, 3-k} \operatorname{sgn}(b)) \eta \left(-x_k, -\frac{r_{02}}{r_{13}} \right) \eta \left(r_{02}, \frac{1}{r_{13}} \right) \Big\}.
 \end{aligned}$$

ε is understood as infinitesimally small.

4.4. Reduction of scalar and tensor integrals for vanishing Gram determinant

Using the four-dimensionality of space-time the scalar five-point function can be reduced to five scalar four-point functions [29, 31]. Furthermore if the Gram determinant of the external momenta of a tensor integral T^N vanishes,

$$|X_{N-1}| = \begin{vmatrix} p_1^2 & p_1 p_2 & \cdots & p_1 p_{N-1} \\ p_2 p_1 & p_2^2 & \cdots & p_2 p_{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ p_{N-1} p_1 & p_{N-1} p_2 & \cdots & p_{N-1}^2 \end{vmatrix} = 0, \tag{4.44}$$

this tensor integral can be expressed in terms of N integrals T^{N-1} . This is always the case for $N \geq 6$, because any five momenta are linear dependent in four dimensions.

4.4.1. Reduction of five-point functions

Here we assume that the four external momenta appearing in the five-point function span the whole four-dimensional space.⁶⁾ Then the integration momentum q depends linearly on these four external momenta and the following equation holds

$$0 = \begin{vmatrix} 2q^2 & 2qp_1 & \cdots & 2qp_4 \\ 2p_1q & 2p_1^2 & \cdots & 2p_1p_4 \\ \vdots & \vdots & \ddots & \vdots \\ 2p_4q & 2p_4p_1 & \cdots & 2p_4^2 \end{vmatrix} = \begin{vmatrix} 2D_0 + Y_{00} & 2qp_1 & \cdots & 2qp_4 \\ D_1 - D_0 + Y_{10} - Y_{00} & 2p_1^2 & \cdots & 2p_1p_4 \\ \vdots & \vdots & \ddots & \vdots \\ D_4 - D_0 + Y_{40} - Y_{00} & 2p_4p_1 & \cdots & 2p_4^2 \end{vmatrix} \tag{4.45}$$

with

$$Y_{ij} = m_i^2 + m_j^2 - (p_i - p_j)^2, \tag{4.46}$$

and D_i as defined in (4.2). Thus we have

$$\frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^Dq \frac{q_{\mu_1} \cdots q_{\mu_P}}{D_0 D_1 \cdots D_4} \begin{vmatrix} 2D_0 + Y_{00} & 2qp_1 & \cdots & 2qp_4 \\ D_1 - D_0 + Y_{10} - Y_{00} & 2p_1^2 & \cdots & 2p_1p_4 \\ \vdots & \vdots & \ddots & \vdots \\ D_4 - D_0 + Y_{40} - Y_{00} & 2p_4p_1 & \cdots & 2p_4^2 \end{vmatrix} = 0. \tag{4.47}$$

⁶⁾ The exceptional case, when they are linear dependent will be covered in the next section.

Expanding the determinant along the first column we obtain

$$\begin{aligned}
 0 = & [2T_{\mu_1 \dots \mu_P}^4(0) + Y_{00} T_{\mu_1 \dots \mu_P}^5] \begin{vmatrix} 2p_1 p_1 & \dots & 2p_1 p_4 \\ \vdots & & \vdots \\ 2p_4 p_1 & \dots & 2p_4 p_4 \end{vmatrix} \\
 & + \sum_{k=1}^4 (-1)^k [T_{\mu_1 \dots \mu_P}^4(k) - T_{\mu_1 \dots \mu_P}^4(0) - p_{4\mu} T_{\mu_1 \dots \mu_P}^4(0) \\
 & + p_{4\mu} T_{\mu_1 \dots \mu_P}^4(0) + (Y_{k0} - Y_{00}) T_{\mu_1 \dots \mu_P}^5] \\
 & \times \begin{vmatrix} 2p_{1\mu} & \dots & 2p_{4\mu} \\ 2p_1 p_1 & \dots & 2p_1 p_4 \\ \vdots & & \vdots \\ 2p_{k-1} p_1 & \dots & 2p_{k-1} p_4 \\ 2p_{k+1} p_1 & \dots & 2p_{k+1} p_4 \\ \vdots & & \vdots \\ 2p_4 p_1 & \dots & 2p_4^2 \end{vmatrix} \tag{4.48}
 \end{aligned}$$

where the arguments of the function T^4 denote again the cancelled propagators.

Since the four momenta p_1, \dots, p_4 span the whole four-dimensional space, the basis of the Lorentz decomposition of the tensor integrals in (4.48) can be chosen such that only tensors built of these momenta but not the metric tensor are involved. $T_{\mu_1 \dots \mu_P}^5(k)$ does not depend on p_k , consequently each term in its Lorentz decomposition contains a factor $p_{i\mu}$, $i \neq k$, and its contraction with the corresponding determinant in (4.48) vanishes. Similarly all terms in the tensor integral decomposition of $T_{\mu_1 \dots \mu_P}^4(0) + p_{4\mu} T_{\mu_1 \dots \mu_P}^4(0)$ involve a factor $p_{i\mu} - p_{4\mu}$, $i=1, 2, 3$, if one performs the shift $q \rightarrow q - p_4$ to bring the tensor integral to the standard form. Multiplying with the determinants and performing the sum in (4.48) these terms drop out. Finally in the term $p_{4\mu} T_{\mu_1 \dots \mu_P}^4(0)$ the determinant is nonzero only for $k=4$ where it can be combined with the first term in (4.48). Rewriting the resulting equation as a determinant and reinserting the explicit form of the tensor integrals we find

$$\frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \frac{q_{\mu_1} \dots q_{\mu_P}}{D_0 D_1 \dots D_4} \begin{vmatrix} D_0 + Y_{00} & 2q p_1 & \dots & 2q p_4 \\ Y_{10} - Y_{00} & 2p_1^2 & \dots & 2p_1 p_4 \\ \vdots & \vdots & & \vdots \\ Y_{40} - Y_{00} & 2p_4 p_1 & \dots & 2p_4^2 \end{vmatrix} = 0. \tag{4.49}$$

Using

$$\begin{aligned}
 2p_i p_j &= Y_{ij} - Y_{i0} - Y_{0j} + Y_{00}, \\
 2q p_j &= D_j - D_0 + Y_{0j} - Y_{00},
 \end{aligned} \tag{4.50}$$

adding the first column to each of the other columns and then enlarging the determinant by one column and one row this can be written as

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$$\begin{vmatrix} 1 & Y_{00} & \dots & Y_{04} \\ 0 & T_{\mu_1 \dots \mu_P}^4(0) + Y_{00} T_{\mu_1 \dots \mu_P}^5 & \dots & T_{\mu_1 \dots \mu_P}^4(4) + Y_{40} T_{\mu_1 \dots \mu_P}^5 \\ 0 & Y_{10} - Y_{00} & \dots & Y_{14} - Y_{04} \\ \vdots & \vdots & & \vdots \\ 0 & Y_{40} - Y_{00} & \dots & Y_{44} - Y_{04} \end{vmatrix} = 0. \quad (4.51)$$

This is equivalent to

$$(4.48) \quad \begin{vmatrix} T_{\mu_1 \dots \mu_P}^5 - T_{\mu_1 \dots \mu_P}^4(0) - T_{\mu_1 \dots \mu_P}^4(1) - T_{\mu_1 \dots \mu_P}^4(2) - T_{\mu_1 \dots \mu_P}^4(3) - T_{\mu_1 \dots \mu_P}^4(4) \\ 1 & Y_{00} & Y_{01} & Y_{02} & Y_{03} & Y_{04} \\ 1 & Y_{10} & Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ 1 & Y_{20} & Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ 1 & Y_{30} & Y_{31} & Y_{32} & Y_{33} & Y_{34} \\ 1 & Y_{40} & Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{vmatrix} = 0, \quad (4.52)$$

which can be solved for $T_{\mu_1 \dots \mu_P}^5$ if the determinant of the matrix Y_{ij} , $i, j = 0, \dots, 4$ is nonzero. Note that in the tensor integral $T_{\mu_1 \dots \mu_P}^4(0)$ the momenta have not been shifted. In particular (4.52) yields the scalar five-point function T_0^5 in terms of five scalar four-point functions.

4.4.2. Reduction of the scalar N-point function for zero Gram determinant

For vanishing Gram determinant $|X_{N-1}|$ the following relation holds, if the Lorentz decomposition of the appearing tensor integrals contains only momenta and no metric tensors, which is the case for $N \geq 5$ or $P = 0$ (scalar integrals)

$$\frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \frac{q_{\mu_1} \dots q_{\mu_P}}{D_0 D_1 \dots D_{N-1}} \begin{vmatrix} D_0 + Y_{00} & 2q p_1 & \dots & 2q p_{N-1} \\ Y_{10} - Y_{00} & 2p_1^2 & \dots & 2p_1 p_{N-1} \\ \vdots & \vdots & & \vdots \\ Y_{N-10} - Y_{00} & 2p_{N-1} p_1 & \dots & 2p_{N-1}^2 \end{vmatrix} = 0. \quad (4.53)$$

Performing the same manipulations of the determinant as in (4.49) to (4.52) above this results in

$$(4.49) \quad \begin{vmatrix} T_{\mu_1 \dots \mu_P}^N - T_{\mu_1 \dots \mu_P}^{N-1}(0) - T_{\mu_1 \dots \mu_P}^{N-1}(1) \dots - T_{\mu_1 \dots \mu_P}^{N-1}(N-1) \\ 1 & Y_{00} & Y_{01} & \dots & Y_{0N-1} \\ 1 & Y_{10} & Y_{11} & \dots & Y_{1N-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & Y_{N-10} & Y_{N-11} & \dots & Y_{N-1N-1} \end{vmatrix} = 0, \quad (4.54)$$

(4.50) valid for $|X_{N-1}| = 0$ and $N \geq 5$ or $P = 0$. We stress again that in the tensor integral $T_{\mu_1 \dots \mu_P}^N(0)$ appearing in (4.54) the momenta have not been shifted. Eq. (4.54) determines $T_{\mu_1 \dots \mu_P}^N$ in terms of $T_{\mu_1 \dots \mu_P}^{N-1}(i)$, $i = 0, \dots, N-1$, if the determinant of the matrix Y_{ij} is nonzero. The vanishing of the determinant corresponds to the leading Landau singularity of T^N which is clearly not contained in T^{N-1} . In this case one has to calculate T^N directly [30].

Eq. (4.54) in particular expresses T_0^N by T_0^{N-1} . For $N=5$ (4.54) coincides with (4.52), which is thus valid for arbitrary momenta. For $N>6$ one can choose any six out of the N denominator factors resulting in different reductions. For $N\leq 4$, where (4.54) is only valid for scalar integrals, the Gram determinant is singular at the edge of phase space where some of the momenta p_i become collinear, i.e. for forward or backward scattering or at the threshold of a certain process. Because in this special situations all integrals can be reduced to lower ones one can obtain considerably simpler formulae than in the general case (see e.g. [33]).

With the methods described in this section all tensor integrals with $N\geq 5$ can be reduced directly to tensor integrals with smaller N . Note that this may yield tensor integrals with $P>N$ because P is not reduced simultaneously as in the reduction method described in Sect. 4.2. These tensor integrals are not directly present in renormalizable theories. Nevertheless their reduction to scalar integrals can be done with the formulae given in Sect. 4.2.

4.5. UV-divergent parts of tensor integrals

For practical calculations it is useful to know the UV-divergent parts of the tensor integrals explicitly. We give directly the products of $D-4$ with all divergent one-loop tensor coefficient integrals appearing in renormalizable theories up to terms of the order $O(D-4)$

$$\begin{aligned}
 (D-4) A_0(m) &= -2m^2, \\
 (D-4) B_0(p_{10}, m_0, m_1) &= -2, \\
 (D-4) B_1(p_{10}, m_0, m_1) &= 1, \\
 (D-4) B_{00}(p_{10}, m_0, m_1) &= \frac{1}{6}(p_{10}^2 - 3m_0^2 - 3m_1^2), \\
 (D-4) B_{11}(p_{10}, m_0, m_1) &= -\frac{2}{3}, \\
 (D-4) C_{00}(p_{10}, p_{20}, m_0, m_1, m_2) &= -\frac{1}{2}, \\
 (D-4) C_{00i}(p_{10}, p_{20}, m_0, m_1, m_2) &= \frac{1}{6}, \\
 (D-4) D_{0000}(p_{10}, p_{20}, p_{30}, m_0, m_1, m_2, m_3) &= -\frac{1}{12}.
 \end{aligned} \tag{4.55}$$

All other scalar coefficients defined in (4.7) and (4.8) are UV-finite.

5. Standard Matrix Elements

5.1. Definition

The invariant matrix elements for scattering and decay processes involving external fermions and/or vector bosons depend on the polarization σ_i, σ'_i and λ_i of these particles. This dependence is completely contained in the polarization vectors $\varepsilon_{\mu_i}(k_i, \lambda_i)$ and spinors $\bar{v}_{\alpha_i}(p'_i, \sigma'_i)$ and $u_{\alpha_i}(p_i, \sigma_i)$. k_i, p'_i and p_i denote the incoming momenta of the vector bosons, antifermions and fermions, respectively. For outgoing fermions one has to replace p, p' by $-p, -p'$ and one must use $u(-p)=v(p)$. If we split off the polarization vectors and spinors from the invariant matrix element \mathcal{M} we are left with a tensor involving Lorentz and Dirac indices in the general case

$$\mathcal{M} = \bar{v}_{\alpha_1}(p'_1, \sigma'_1) \dots \bar{v}_{\alpha_n}(p'_n, \sigma'_n) \mathcal{M}_{\alpha_1 \dots \alpha_n \beta_1 \dots \beta_n}^{\mu_1 \dots \mu_m} u_{\beta_1}(p_1, \sigma_1) \dots u_{\beta_n}(p_n, \sigma_n) \times \varepsilon_{\mu_1}(k_1, \lambda_1) \dots \varepsilon_{\mu_m}(k_m, \lambda_m). \tag{5.1}$$

To be definite we choose m external vector bosons and n external fermion-antifermion pairs. The tensor $\mathcal{M}_{\alpha_1 \dots \alpha_n \beta_1 \dots \beta_n}^{\mu_1 \dots \mu_m}$ can be decomposed into a set of covariant operators together with the corresponding scalar formfactors F_i

$$\mathcal{M}_{\alpha_1 \dots \alpha_n \beta_1 \dots \beta_n}^{\mu_1 \dots \mu_m} = \sum_i \mathcal{M}_{i, \alpha_1 \dots \alpha_n \beta_1 \dots \beta_n}^{\mu_1 \dots \mu_m} F_i. \tag{5.2}$$

We call the covariant operators $\mathcal{M}_{i, \alpha_1 \dots \alpha_n \beta_1 \dots \beta_n}^{\mu_1 \dots \mu_m}$ multiplied by the corresponding polarization vectors and spinors standard matrix elements \mathcal{M}_i

$$\mathcal{M}_i = \bar{v}_{\alpha_1}(p'_1, \sigma'_1) \dots \bar{v}_{\alpha_n}(p'_n, \sigma'_n) \mathcal{M}_{i, \alpha_1 \dots \alpha_n \beta_1 \dots \beta_n}^{\mu_1 \dots \mu_m} u_{\beta_1}(p_1, \sigma_1) \dots u_{\beta_n}(p_n, \sigma_n) \times \varepsilon_{\mu_1}(k_1, \lambda_1) \dots \varepsilon_{\mu_m}(k_m, \lambda_m). \tag{5.3}$$

In this way the invariant amplitude \mathcal{M} is decomposed into polarization independent formfactors F_i and the standard matrix elements \mathcal{M}_i

$$\mathcal{M} = \sum_i \mathcal{M}_i F_i. \tag{5.4}$$

The formfactors F_i are complicated model dependent functions involving in general the invariant integrals T^N and the counterterms. The standard matrix elements in contrast are simple model independent expressions which depend on the external particles only but contain the whole information on their polarization. They are purely kinematical objects. All of the dynamical information is contained in the formfactors.

The covariant tensor operators forming the standard matrix elements can be constructed from the external four-momenta p_i, p'_i and k_i , the Lorentz tensors $g^{\mu\nu}$ and $e^{\mu\nu\varrho\sigma}$ and the Dirac matrices γ^μ, γ_5 . In general one thus obtains an overcomplete set. Dirac algebra and momentum conservation are used to eliminate superfluous operators. Since the external particles are on-shell, the Dirac equation for the fermion spinors

$$\not{p}_i u(p_i, \sigma_i) = m_i u(p_i, \sigma_i), \quad \bar{v}(p'_i, \sigma'_i) \not{p}'_i = -m'_i \bar{v}(p'_i, \sigma'_i) \tag{5.5}$$

and the transversality condition for the polarization vectors

$$k_i^{\mu_i} \varepsilon_{\mu_i}(k_i, \lambda_i) = 0 \tag{5.6}$$

reduce the number of independent standard matrix elements further.

The number of independent standard matrix elements cannot be larger than the number of independent polarization combinations of the external particles. In four dimensions there are only four linear independent four-vectors. Expressing all four-vectors in a definite basis allows to derive the missing relations between the remaining standard matrix elements. Thus a minimal set of standard matrix elements can be constructed.

If there are only few external particles there may be less independent standard matrix elements than different polarization combinations, since there are only few momenta available for their construction. In this case some of the polarized amplitudes are related.

The number of standard matrix elements can be reduced further if the model under consideration exhibits certain symmetries. These evidently also apply to the relevant standard matrix elements.

For many applications it is not essential to minimize the number of standard matrix elements. All one needs is a complete set.

Furthermore the choice of the standard matrix elements is not unique. This allows to arrange for the most convenient set according to simplicity, the structure of the lowest order amplitudes and, if present, symmetries. At least some of the formfactors can be chosen as generalizations of the lowest order couplings. This is useful in establishing improved Born approximations.

The concept of standard matrix elements is not indispensable for the calculation of amplitudes in higher orders. It is, however, extremely helpful in organizing lengthy calculations, which often are inevitable. All complicated expressions are cast into the formfactors which are polarization independent and thus have to be evaluated only once.

An alternative method would be to calculate directly the polarized amplitudes. This requires a definite representation for the spinors and/or polarization vectors from the start. The whole calculation has to be done for each polarization separately. A closer look shows that this method can be represented as a particular case of the standard matrix element approach. The corresponding covariant operators are constructed from the polarization vectors and spinors instead of the momenta, Lorentz tensors and Dirac matrices. Their explicit form is

$$\mathcal{M}_{i_1, \lambda_1 \dots i_n, \lambda_n}^{\mu_1 \dots \mu_m} = (-1)^n \frac{v_{\alpha_1}(p_1, \sigma_1)}{2m_1'} \dots \frac{v_{\alpha_n}(p_n, \sigma_n)}{2m_n'} \frac{\bar{u}_{\beta_1}(p_1, \sigma_1)}{2m_1} \dots \frac{\bar{u}_{\beta_n}(p_n, \sigma_n)}{2m_n} \times \epsilon_{\mu_1}^*(k_1, \lambda_1) \dots \epsilon_{\mu_m}^*(k_m, \lambda_m), \tag{5.7}$$

where m, m' are the masses of the external spinors. The indices i correspond to different polarization combinations. Consequently the number of different standard matrix elements equals the number of polarizations of the external particles. For each polarization only one standard matrix element is nonzero. In this sense the set of standard matrix elements (5.7) is orthogonal. The formfactors equal the polarization amplitudes and are directly obtained by inserting explicitly the polarization vectors and spinors in the invariant matrix element. Unlike in the approach outlined above these formfactors are no direct generalizations of the lowest order couplings.

In the following we list complete sets of standard matrix elements relevant for the production of bosons in fermion-antifermion annihilation.

5.2. Standard matrix elements for processes with two external fermions

In this section we will give the standard matrix elements for processes involving two external fermions ($F\bar{F}$) and one [or two] scalar (S) or vector (V) bosons. The momenta and spinors of the fermions are denoted by p_1, p_2 and $\bar{v}(p_1) = \bar{v}(p_1, \sigma_1), u(p_2) = u(p_2, \sigma_2)$, the momenta and polarization vectors of the bosons by $k_1, \epsilon_1 = \epsilon(k_1, \lambda_1)$ [and $k_2, \epsilon_2 = \epsilon(k_2, \lambda_2)$]. The numbers of different polarizations for each scalar, fermion and vector boson are 1, 2 and 3, respectively. If we use momentum conservation to eliminate k_1 [or $k_1 + k_2$] the standard matrix elements are constructed from the momenta p_2 and p_1 [and $k_1 - k_2$], the polarization vectors of the vector bosons, the totally antisymmetric tensor $\epsilon^{\mu\nu\rho\sigma}$ and Dirac matrices between the spinors. If there are products of ϵ -tensors, pairs of them can be eliminated using

$$\epsilon^{\mu\nu\rho\sigma} \epsilon^{\alpha\beta\gamma\delta} = - \begin{vmatrix} g^{\mu\alpha} & g^{\mu\beta} & g^{\mu\gamma} & g^{\mu\delta} \\ g^{\nu\alpha} & g^{\nu\beta} & g^{\nu\gamma} & g^{\nu\delta} \\ g^{\rho\alpha} & g^{\rho\beta} & g^{\rho\gamma} & g^{\rho\delta} \\ g^{\sigma\alpha} & g^{\sigma\beta} & g^{\sigma\gamma} & g^{\sigma\delta} \end{vmatrix}. \tag{5.8}$$

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If any of the left over ε -tensors are contracted with four four-momenta, we write for one of these momenta $p^\alpha = \frac{1}{2}\{\not{p}, \gamma^\alpha\}$ between the spinors. Now all remaining ε -tensors are contracted with one γ -matrix at least and can be eliminated using the Chisholm identity⁷⁾

$$\varepsilon_{\mu\nu\rho\sigma}\gamma^\sigma = -i[\gamma_\mu\gamma_\nu\gamma_\rho - g_{\mu\nu}\gamma_\rho + g_{\mu\rho}\gamma_\nu - g_{\nu\rho}\gamma_\mu]\gamma_5. \tag{5.9}$$

All Dirac matrices contracted with p_1 or p_2 can be eliminated using Dirac algebra and the Dirac equation. Consequently the only remaining Dirac matrices are contracted with polarization vectors [and $\not{k}_1 - \not{k}_2$] and there is at most one of each type. Finally in the scalar products involving the polarization vectors only one [or two] independent momenta may appear because of transversality and momentum conservation.

Thus we arrive at the following sets of standard matrix elements (we suppress polarization indices in the following):

5.2.1. $S \rightarrow F\bar{F}$

There are $2 \times 2 = 4$ different polarization combinations but only two standard matrix elements

$$\mathcal{M}^\sigma = \bar{u}(p_1)\omega_\sigma v(p_2), \tag{5.10}$$

where $\sigma = \pm$ and $\omega_\pm = \frac{1 \pm \gamma_5}{2}$ and the fermions are outgoing.

5.2.2. $V \rightarrow F\bar{F}$

Replacing the scalar by a vector results in $3 \times 2 \times 2 = 12$ different polarizations and yet only four standard matrix elements

$$\begin{aligned} \mathcal{M}_1^\sigma &= \bar{u}(p_1)\not{\epsilon}_1\omega_\sigma v(p_2), \\ \mathcal{M}_2^\sigma &= \bar{u}(p_1)\omega_\sigma v(p_2)\epsilon_1 p_1. \end{aligned} \tag{5.11}$$

5.2.3. $F\bar{F} \rightarrow SS$

Here the number of independent polarizations four equals the number of standard matrix elements

$$\begin{aligned} \mathcal{M}_1^\sigma &= \bar{v}(p_1)\frac{1}{2}(\not{k}_1 - \not{k}_2)\omega_\sigma u(p_2), \\ \mathcal{M}_2^\sigma &= \bar{v}(p_1)\omega_\sigma u(p_2). \end{aligned} \tag{5.12}$$

5.2.4. $F\bar{F} \rightarrow SV$

In this case we find twelve standard matrix elements for $2 \times 2 \times 3 = 12$ different polarizations

⁷⁾ Eq. (5.8) and (5.9) can be applied because the standard matrix elements involve only external vectors and spinors which remain four-dimensional also in dimensional regularization.

$$\begin{aligned}
\mathcal{M}_1^\sigma &= \bar{v}(p_1) \not{\epsilon}_2 \omega_\sigma u(p_2), \\
\mathcal{M}_2^\sigma &= \bar{v}(p_1) \frac{1}{2}(\not{k}_1 - \not{k}_2) \omega_\sigma u(p_2) \epsilon_2 p_1, \\
\mathcal{M}_3^\sigma &= \bar{v}(p_1) \frac{1}{2}(\not{k}_1 - \not{k}_2) \omega_\sigma u(p_2) \epsilon_2 p_2, \\
\mathcal{M}_4^\sigma &= \bar{v}(p_1) \not{\epsilon}_2 \not{k}_2 \omega_\sigma u(p_2), \\
\mathcal{M}_5^\sigma &= \bar{v}(p_1) \omega_\sigma u(p_2) \epsilon_2 p_1, \\
\mathcal{M}_6^\sigma &= \bar{v}(p_1) \omega_\sigma u(p_2) \epsilon_2 p_2.
\end{aligned} \tag{5.13}$$

5.2.5. $F\bar{F} \rightarrow VV$

There are $2 \times 2 \times 3 \times 3 = 36$ different polarization combinations, however, we can construct 40 standard matrix elements

$$\begin{aligned}
\mathcal{M}_1^\sigma &= \bar{v}(p_1) \not{\epsilon}_1 (\not{k}_1 - \not{p}_1) \not{\epsilon}_2 \omega_\sigma u(p_2), \\
\mathcal{M}_2^\sigma &= \bar{v}(p_1) \frac{1}{2}(\not{k}_1 - \not{k}_2) (\epsilon_1 \epsilon_2) \omega_\sigma u(p_2), \\
\mathcal{M}_{3,1}^\sigma &= \bar{v}(p_1) \not{\epsilon}_1 \omega_\sigma u(p_2) (\epsilon_2 k_1), \\
\mathcal{M}_{3,2}^\sigma &= -\bar{v}(p_1) \not{\epsilon}_2 \omega_\sigma u(p_2) (\epsilon_1 k_2), \\
\mathcal{M}_{4,1}^\sigma &= \bar{v}(p_1) \not{\epsilon}_1 \omega_\sigma u(p_2) (\epsilon_2 p_2), \\
\mathcal{M}_{4,2}^\sigma &= -\bar{v}(p_1) \not{\epsilon}_2 \omega_\sigma u(p_2) (\epsilon_1 p_1), \\
\mathcal{M}_5^\sigma &= \bar{v}(p_1) \frac{1}{2}(\not{k}_1 - \not{k}_2) \omega_\sigma u(p_2) (\epsilon_1 k_2) (\epsilon_2 k_1), \\
\mathcal{M}_6^\sigma &= \bar{v}(p_1) \frac{1}{2}(\not{k}_1 - \not{k}_2) \omega_\sigma u(p_2) (\epsilon_1 p_1) (\epsilon_2 p_2), \\
\mathcal{M}_{7,1}^\sigma &= \bar{v}(p_1) \frac{1}{2}(\not{k}_1 - \not{k}_2) \omega_\sigma u(p_2) (\epsilon_1 k_2) (\epsilon_2 p_2), \\
\mathcal{M}_{7,2}^\sigma &= \bar{v}(p_1) \frac{1}{2}(\not{k}_1 - \not{k}_2) \omega_\sigma u(p_2) (\epsilon_1 p_1) (\epsilon_2 k_1), \\
\mathcal{M}_{11}^\sigma &= \bar{v}(p_1) \not{\epsilon}_1 \not{\epsilon}_2 \omega_\sigma u(p_2), \\
\mathcal{M}_{12}^\sigma &= \bar{v}(p_1) \omega_\sigma u(p_2) (\epsilon_1 \epsilon_2), \\
\mathcal{M}_{13,1}^\sigma &= \bar{v}(p_1) \not{\epsilon}_1 \not{k}_1 \omega_\sigma u(p_2) (\epsilon_2 k_1), \\
\mathcal{M}_{13,2}^\sigma &= \bar{v}(p_1) \not{k}_2 \not{\epsilon}_2 \omega_\sigma u(p_2) (\epsilon_1 k_2), \\
\mathcal{M}_{14,1}^\sigma &= \bar{v}(p_1) \not{\epsilon}_1 \not{k}_1 \omega_\sigma u(p_2) (\epsilon_2 p_2), \\
\mathcal{M}_{14,2}^\sigma &= \bar{v}(p_1) \not{k}_2 \not{\epsilon}_2 \omega_\sigma u(p_2) (\epsilon_1 p_1), \\
\mathcal{M}_{15}^\sigma &= \bar{v}(p_1) \omega_\sigma u(p_2) (\epsilon_1 k_2) (\epsilon_2 k_1), \\
\mathcal{M}_{16}^\sigma &= \bar{v}(p_1) \omega_\sigma u(p_2) (\epsilon_1 p_1) (\epsilon_2 p_2), \\
\mathcal{M}_{17,1}^\sigma &= \bar{v}(p_1) \omega_\sigma u(p_2) (\epsilon_1 k_2) (\epsilon_2 p_2), \\
\mathcal{M}_{17,2}^\sigma &= \bar{v}(p_1) \omega_\sigma u(p_2) (\epsilon_1 p_1) (\epsilon_2 k_1).
\end{aligned} \tag{5.14}$$

We have obtained more than 36 standard matrix elements because we have not yet used the four dimensionality of space time, i.e. the fact that the five vectors $p_1, p_2, k_1 - k_2, \epsilon_1$ and ϵ_2 are linear dependent. The relations between the 40 standard matrix elements can be found by representing these vectors in a certain basis using for example $v_1 = p_1 + p_2, v_2 = p_1 - p_2, v_3 = k_1 - k_2, v_{4,\mu} = \epsilon_{\mu\nu\rho\sigma} v_1^\nu v_2^\rho v_3^\sigma$. In this way one can derive the relation

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$$\begin{aligned}
 0 = & 2(p_1 p_2 - m_1 m_2) (\mathcal{M}_1^\sigma + \mathcal{M}_2^\sigma) \\
 & - 2(p_2 k_2 - m_2^2 - m_1 m_2) \mathcal{M}_{3,1}^\sigma - 2(p_1 k_1 - m_1^2 - m_1 m_2) \mathcal{M}_{3,2}^\sigma \\
 & - 2(k_1^2 + (k_1 k_2)) \mathcal{M}_{4,1}^\sigma - 2(k_2^2 + (k_1 k_2)) \mathcal{M}_{4,2}^\sigma - 2\mathcal{M}_5^\sigma + 2(\mathcal{M}_{7,1}^\sigma + \mathcal{M}_{7,2}^\sigma) \\
 & - 2(m_1(m_2^2 - p_2 k_2) + m_2(m_1^2 - p_1 k_1)) (\mathcal{M}_{11}^\sigma - \mathcal{M}_{12}^\sigma) \\
 & + (m_1 + m_2) (p_1 p_2 - m_1 m_2) \mathcal{M}_{12}^\sigma - 2m_2 \mathcal{M}_{13,1}^\sigma - 2m_1 \mathcal{M}_{13,2}^\sigma \\
 & + (m_1 + m_2) (2\mathcal{M}_{14,1}^\sigma + 2\mathcal{M}_{14,2}^\sigma - \mathcal{M}_{15}^\sigma - 4\mathcal{M}_{16}^\sigma) \\
 & + (3m_1 + m_2) \mathcal{M}_{17,1}^\sigma + (3m_2 + m_1) \mathcal{M}_{17,2}^\sigma
 \end{aligned} \tag{5.15}$$

and a similar independent one allowing to eliminate four of the 40 standard matrix elements (5.14).

The construction of complete sets of standard matrix elements described above is straightforward. The reduction of general structures to these standard matrix elements is therefore easy to implement into computer algebra programs. In practical applications some of the standard matrix elements may not contribute due to the presence of symmetries and/or the neglect of fermion masses. These aspects will be discussed together with the applications in the following chapters.

5.3. Calculation of standard matrix elements

For the calculation of the standard matrix elements one has to choose a certain representation for the polarization vectors and spinors. This has to be done only once for each process and not in the calculation of individual Feynman diagrams. If there are at least four external particles the polarization vectors can be constructed from their four-momenta respecting

$$\begin{aligned}
 k_i \cdot \varepsilon_i(k_i, \lambda_i) &= 0, \\
 \varepsilon_i(k_i, \lambda_i) \varepsilon_i(k_i, \lambda'_i) &= -\delta_{\lambda_i, \lambda'_i}.
 \end{aligned} \tag{5.16}$$

We thus obtain for ε_2

$$\begin{aligned}
 \varepsilon_2^\mu(k_2, \parallel) &= \frac{1}{\sqrt{p_1 p_2 (2p_1 k_2 p_2 k_2 - k_2^2 p_1 p_2) + p_1^2 p_2^2 k_2^2 - p_2^2 (p_1 k_2)^2 - p_1^2 (p_2 k_2)^2}} \\
 &\quad \times \frac{1}{\sqrt{(p_2 k_2 + p_1 k_2)^2 - k_2^2 (p_1 + p_2)^2}} \\
 &\quad \times [p_2^\mu (p_1 (p_1 + p_2) k_2^2 - p_1 k_2 (p_2 k_2 + p_1 k_2)) \\
 &\quad - p_1^\mu (p_2 (p_1 + p_2) k_2^2 - p_2 k_2 (p_2 k_2 + p_1 k_2)) \\
 &\quad + k_2^\mu (p_1 p_2 (p_1 k_2 - p_2 k_2) + p_2^2 p_1 k_2 - p_1^2 p_2 k_2)] \\
 &= (0, \cos \vartheta, 0, -\sin \vartheta)
 \end{aligned} \tag{5.17}$$

$$\begin{aligned}
 \varepsilon_2^\mu(k_2, \perp) &= \frac{1}{\sqrt{p_1 p_2 (2p_1 k_2 p_2 k_2 - k_2^2 p_1 p_2) + p_1^2 p_2^2 k_2^2 - p_2^2 (p_1 k_2)^2 - p_1^2 (p_2 k_2)^2}} \varepsilon_{\nu\alpha\sigma}^\mu p_2^\nu p_1^\alpha k_2^\sigma \\
 &= (0, 0, 1, 0)
 \end{aligned} \tag{5.18}$$

$$\begin{aligned} \varepsilon_2^\mu(k_2, L) &= \frac{1}{\sqrt{k_2^2 [(p_2 k_2 + p_1 k_2)^2 - k_2^2 (p_2 + p_1)^2]}} [k_2^\mu (p_2 k_2 + p_1 k_2) - (p_1 + p_2)^\mu k_2^2] \\ &= (k, E_2 \sin \vartheta, 0, E_2 \cos \vartheta) / \sqrt{k_2^2}, \end{aligned} \tag{5.19}$$

where we have also given the simple expressions in the CMS-system of the fermions and bosons. In this system the four-momenta of the external particles read

$$\begin{aligned} p_{1,2} &= (\tilde{E}_{1,2}, 0, 0, \mp |\mathbf{p}|), \\ k_{1,2} &= (E_{1,2}, \mp |\mathbf{k}| \sin \vartheta, 0, \mp |\mathbf{k}| \cos \vartheta). \end{aligned} \tag{5.20}$$

$\tilde{E}_{1,2}$ are the energies and \mathbf{p} the three-momentum of the fermions and $E_{1,2}$ the energies and \mathbf{k} the three-momentum of the bosons. ϑ is the angle between the spatial vectors \mathbf{p} and \mathbf{k} .

From the polarization vectors given above the ones for helicity states are obtained as

$$\varepsilon_2^\mu(k_2, \pm) = \frac{1}{\sqrt{2}} [\varepsilon_2^\mu(k_2, \parallel) \pm i \varepsilon_2^\mu(k_2, \perp)], \quad \varepsilon_2^\mu(k_2, 0) = \varepsilon_2^\mu(k_2, L). \tag{5.21}$$

The polarization vector ε_1 can be obtained by interchanging $1 \leftrightarrow 2$.

For the case of only three external particles one needs a further independent vector. It can be chosen freely but linear independent of the momenta. Using this additional vector as one of the polarization vectors the others can be constructed using (5.16).

Inserting the polarization vectors (5.19) into the standard matrix elements these can be reduced to the ones for external scalars, i.e. to (5.10) for the decay $V \rightarrow F\bar{F}$, and to (5.12) for the annihilation processes $F\bar{F} \rightarrow VV, VS$. To calculate these remaining Dirac matrix elements one either inserts a definite representation for the spinors or evaluates the quantities $\mathcal{M}_i^{\sigma*} \mathcal{M}_j^{\sigma'}$ via traces and reconstructs \mathcal{M}_i^σ from those if needed. Note that for the calculation of $|\mathcal{M}|^2$ to one-loop order one only has to evaluate the products $\mathcal{M}_i^{\sigma*} \mathcal{M}_j^\sigma$ for those values of i , where F_i^σ is nonzero in lowest order

$$\begin{aligned} |\mathcal{M}|^2 &= |\mathcal{M}_0 + \delta \mathcal{M}_1|^2 \approx |\mathcal{M}_0|^2 + 2 \operatorname{Re} \{ \mathcal{M}_0^* \delta \mathcal{M}_1 \} \\ &= \sum_{i,j} F_{i,0}^{\sigma*} (F_{j,0}^\sigma + 2 \delta F_{j,1}^\sigma) \operatorname{Re} \{ \mathcal{M}_i^{\sigma*} \mathcal{M}_j^\sigma \}. \end{aligned} \tag{5.22}$$

Here $\mathcal{M}_0, F_{i,0}^\sigma$ denote the lowest order quantities and $\delta \mathcal{M}_1, \delta F_{i,1}^\sigma$ the one-loop quantities.

For massless external fermions the Dirac matrix elements (5.10) and (5.12) are equivalent to the helicity matrix elements. They do not interfere and can thus easily be obtained from $|\mathcal{M}_i^\sigma|^2$ as

$$\begin{aligned} \bar{v}(p_1) \omega_\sigma u(p_2) &= \sqrt{2 p_1 p_2}, \\ \bar{v}(p_1) (\not{k}_1 - \not{k}_2) \omega_\sigma u(p_2) &= \sqrt{4 p_1 (k_1 - k_2) p_2 (k_1 - k_2) - 2 p_1 p_2 (k_1 - k_2)^2}. \end{aligned} \tag{5.23}$$

If one is only interested in unpolarized quantities it suffices to calculate $\Sigma_{\text{pol}} \mathcal{M}_i^{\sigma*} \mathcal{M}_j^\sigma$ using the polarization sums for vector bosons and spinors.

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6. Calculation of One-Loop Amplitudes

We have described all the ingredients necessary for the calculation of one-loop radiative corrections. This chapter shows how one-loop amplitudes are evaluated in practice.

First one has to specify a Lagrangian and to derive the corresponding Feynman rules. Then renormalization has to be carried out and the counterterms have to be determined. Both were done in Chap. 2 and 3 for the SM. Once the Feynman rules and the counterterms are fixed, the following steps apply to any renormalizable model.

To calculate the amplitude of a certain process at the one-loop level one has to construct all tree and one-loop Feynman diagrams with the given external particles allowed by the specified model. Next each Feynman diagram has to be reduced algebraically to a form suitable for numerical evaluation. This procedure is explained in more detail in Sect. 6.1. Finally the expressions for all diagrams have to be put together into a numerical program which calculates the amplitude and the corresponding cross section or decay rate.

6.1. Algebraic reduction of Feynman diagrams

The algorithm for the reduction of one-loop diagrams is the following. The loop integral obtained from the Feynman rules contains a product of propagators as denominator and a numerator composed of Lorentz vectors and tensors, Dirac matrices and spinors and polarization vectors of the external particles. The numerator is simplified using tensor and Dirac algebra, the mass shell conditions for the external particles and momentum conservation. One can also try to separate terms proportional to one or more of the denominators. Cancelling these yields N-point functions of lower degree. Next the loop integral is organized into the tensor integrals defined in Sect. 4.1. The Lorentz decomposition of these integrals is inserted and the whole Dirac and Lorentz structure is separated off from the integrals. Using again Dirac algebra and mass shell conditions it can be reduced to the appropriate standard matrix elements as discussed in Chap. 5.

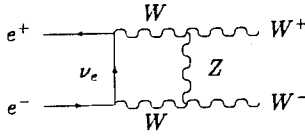
We thus arrive at an expression of the form

$$\delta \mathcal{M} = \sum_i \mathcal{M}_i \delta F_i \quad (6.1)$$

for each one-loop Feynman diagram. The formfactors are linear combinations of the invariant coefficient functions of the tensor integrals with coefficients being functions of the kinematical invariants.

The formfactors can be further evaluated by applying the reduction scheme for the invariant integrals described in Sects. 4.2 and 4.4. Finally they are obtained as linear combinations of the scalar one-loop integrals A_0 , B_0 , C_0 and D_0 which are given explicitly in Sect. 4.3. This last step may lead to very lengthy expressions. Their algebraic evaluation needs a lot of time and space. This can be avoided by performing the reduction to scalar integrals numerically.

The evaluation of the counterterm diagrams and the Born diagrams is done in a similar way. Since no integrations have to be performed their calculation is much easier. In most cases the counterterm diagrams can be obtained from the Born diagrams by replacing the lowest order couplings by the corresponding counterterm.

Fig. 6.1. Box diagram contributing to $e^+e^- \rightarrow W^+W^-$

As an illustration of the reduction method we present the explicit calculation of a box diagram contributing to $e^+e^- \rightarrow W^+W^-$ (Fig. 6.1). According to the Feynman rules the corresponding contribution to the invariant matrix element $\delta\mathcal{M}$ is given by (we include a global factor i in the Feynman rules in order to obtain real amplitudes)

$$\delta\mathcal{M} = -i \frac{e^4 c_W^2}{2s_W^4} \mu^{4-D} \int \frac{d^D q}{(2\pi)^D} \frac{\bar{v}(p_1) \gamma^\mu (q + k_1 - \not{p}_1) \gamma^\nu \omega_- u(p_2) \Gamma_{\lambda\mu\rho} \Gamma^\lambda_{\sigma\nu} \epsilon_1^\rho \epsilon_2^\sigma}{[q^2 - M_Z^2] [(q + k_1)^2 - M_W^2] (q + k_1 - p_1)^2 [(q - k_2)^2 - M_W^2]}, \quad (6.2)$$

with

$$\begin{aligned} \Gamma_{\lambda\mu\rho} &= g_{\lambda\mu} (2q + k_1)_\rho + g_{\mu\rho} (-q - 2k_1)_\lambda + g_{\rho\lambda} (k_1 - q)_\mu, \\ \Gamma_{\lambda\sigma\nu} &= g_{\lambda\sigma} (-k_2 - q)_\nu + g_{\sigma\nu} (2k_2 - q)_\lambda + g_{\nu\lambda} (2q - k_2)_\sigma. \end{aligned} \quad (6.3)$$

Evaluating the numerator and introducing the tensor integrals D and C we arrive at

$$\begin{aligned} \delta\mathcal{M} &= \frac{\alpha^2 c_W^2}{2s_W^4} \bar{v}(p_1) \{ -8 \gamma^\mu \epsilon_1^\nu \epsilon_2^\rho D_{\mu\nu\rho} \\ &+ D_{\mu\nu} [\not{\epsilon}_1 \gamma^\mu \not{\epsilon}_2 2(k_1 - k_2)^\nu \\ &+ \not{\epsilon}_1 (-2\epsilon_2^\mu k_2^\nu - 8p_1^\mu \epsilon_2^\nu + 2p_2^\mu \epsilon_2^\nu) + \not{\epsilon}_1 \gamma^\mu \not{k}_2 8\epsilon_2^\nu \\ &- \not{\epsilon}_2 (-2\epsilon_1^\mu k_1^\nu - 8p_2^\mu \epsilon_1^\nu + 2p_1^\mu \epsilon_1^\nu) - \not{k}_1 \gamma^\mu \not{\epsilon}_2 8\epsilon_1^\nu \\ &+ \not{k}_1 8\epsilon_1^\mu \epsilon_2^\nu + \gamma^\mu (-16\epsilon_1^\nu p_2 \epsilon_2 + 16\epsilon_2^\nu p_1 \epsilon_1 + 2(k_1 - p_1)^\nu \epsilon_1 \epsilon_2) \\ &+ D_\mu [\not{\epsilon}_1 (\not{k}_1 - \not{p}_1) \not{\epsilon}_2 2(k_1 - k_2)^\mu + \not{\epsilon}_1 \gamma^\mu \not{\epsilon}_2 (M_Z^2 - 4k_1 k_2) \\ &+ \not{\epsilon}_1 (-\epsilon_2^\mu (3M_Z^2 + 3M_W^2 + 2p_1 k_1) - 4k_2^\mu \epsilon_2 k_1 + 4p_2^\mu \epsilon_2 k_1) \\ &- \not{\epsilon}_2 (\epsilon_1^\mu (3M_Z^2 + 3M_W^2 + 2p_2 k_2) + 4k_1^\mu \epsilon_1 k_2 - 4p_1^\mu \epsilon_1 k_2) \\ &+ \not{k}_1 (2(k_1 - p_1)^\mu \epsilon_1 \epsilon_2 - 8\epsilon_1^\mu \epsilon_2 p_2 + 8\epsilon_2^\mu \epsilon_1 p_1) \\ &+ \gamma^\mu \epsilon_1 \epsilon_2 (M_Z^2 - 3M_W^2 + 6p_1 k_1) + \not{k}_1 \gamma^\mu \not{\epsilon}_2 4\epsilon_1 k_2 + 4\not{\epsilon}_1 \gamma^\mu \not{k}_2 4\epsilon_2 k_1 \\ &+ D_0 [\not{\epsilon}_1 (\not{k}_1 - \not{p}_1) \not{\epsilon}_2 (M_Z^2 - 4k_1 k_2) + \not{k}_1 \epsilon_1 \epsilon_2 (M_W^2 - 2p_1 k_1 - 3M_Z^2) \\ &+ \not{\epsilon}_1 \epsilon_2 k_1 (4p_2 k_2 - 2M_W^2 + 2M_Z^2) - \not{\epsilon}_2 \epsilon_1 k_2 (4p_1 k_1 - 2M_W^2 + 2M_Z^2)] \\ &+ C_\mu [\not{\epsilon}_1 \gamma^\mu \not{\epsilon}_2 - \not{\epsilon}_2 3\epsilon_1^\mu - \not{\epsilon}_1 3\epsilon_2^\mu + \gamma^\mu \epsilon_1 \epsilon_2] \\ &+ C_0 [\not{\epsilon}_1 2\epsilon_2 k_1 - \not{\epsilon}_2 2\epsilon_1 k_2 + \not{\epsilon}_1 3\epsilon_2 p_2 - \not{\epsilon}_2 3\epsilon_1 p_1 - \not{k}_1 4\epsilon_1 \epsilon_2] \} \omega_- u(p_2). \quad (6.4) \end{aligned}$$

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The three-point integrals arise from q^2 terms in the numerator by writing $q^2 = (q^2 - M_Z^2) + M_Z^2$ and cancelling the first denominator factor. After that the shift $q \rightarrow q - k_1 + p_1$ was performed in the three-point integrals (this shift conserves the manifest CP symmetry of the diagram). The arguments of the C and D functions are as follows

$$\begin{aligned} D &= D(k_1, k_1 - p_1, -k_2, M_Z, M_W, 0, M_W), \\ C &= C(p_1, -p_2, 0, M_W, M_W). \end{aligned} \tag{6.5}$$

Inserting the tensor integral decomposition eqs. (4.7), (4.8) yields the final expression

$$\begin{aligned} \delta \mathcal{M} &= \frac{\alpha^2 c_W^2}{2s_W^4} \{ \mathcal{H}_1^- [20D_{00} + 2(4M_W^2 - s)D_{33} + 2(M_W^2 + t)D_{22} \\ &\quad + (12M_W^2 + 4t - 2s)D_{23} + 2(4M_W^2 - s)D_{13} + (16M_W^2 - 6s + 2M_Z^2)D_3 \\ &\quad + (2t - 2s + 6M_W^2 + M_Z^2)D_2 + (4M_W^2 - 2s + M_Z^2)D_0] \\ &\quad + \mathcal{H}_2^- [-4C_0 - 16D_{003} - 8D_{002} + 10D_{00} + 2tD_{22} + 2(M_W^2 + t)(D_{33} + D_{13}) \\ &\quad + 2(M_W^2 + 3t)D_{23} + 2(M_W^2 - 2t + M_Z^2)D_3 + (M_Z^2 - t)D_2 + (t - 3M_Z^2)D_0] \\ &\quad + [\mathcal{H}_{3,1}^- + \mathcal{H}_{3,2}^-] [-3C_2 + 2C_0 - 8D_{003} - 8D_{00} + (4s - 11M_W^2 + 5t)D_{13} \\ &\quad - 3(M_W^2 + t)D_{33} - 2(2M_W^2 + t)D_{23} + (t - 4M_W^2 - 3M_Z^2)D_3 + 2(M_Z^2 - t)D_0] \\ &\quad + [\mathcal{H}_{4,1}^- + \mathcal{H}_{4,2}^-] [4C_2 + 3C_0 - 8D_{002} - 26D_{00} + 2(s - 4M_W^2)(D_{33} + D_{13}) \\ &\quad - 2(t + 2M_W^2)D_{22} + (4s - 2t - 18M_W^2)D_{23} \\ &\quad + (4s - 8M_W^2 - 2M_Z^2)D_3 + (t - 4M_W^2 - 3M_Z^2)D_2] \\ &\quad + \mathcal{H}_5^- [16D_{113} + 8D_{123} - 8D_{13}] \\ &\quad + \mathcal{H}_6^- [8D_{222} + 16D_{223} + 24D_{22} + 32D_{23} + 16D_2] \\ &\quad + [\mathcal{H}_{7,1}^- + \mathcal{H}_{7,2}^-] [8D_{332} + 8D_{223} + 8D_{123} + 8D_{23} + 16D_{13}] \}, \end{aligned} \tag{6.6}$$

where we introduced the standard matrix elements (5.14), the Mandelstam variables

$$s = (p_1 + p_2)^2, \quad t = (p_1 - k_1)^2, \tag{6.7}$$

and put

$$k_1^2 = k_2^2 = M_W^2, \quad p_1^2 = p_2^2 = 0. \tag{6.8}$$

Furthermore we made use of the relations which follow from the symmetry of the diagram under the exchange $e^+ \leftrightarrow e^-, W^+ \leftrightarrow W^-$ (CP invariance)

$$\begin{aligned} D_1 &= D_3, & D_{11} &= D_{33}, & D_{12} &= D_{23}, \\ D_{001} &= D_{003}, & D_{112} &= D_{332}, & D_{122} &= D_{223}, \\ D_{111} &= D_{333}, & D_{113} &= D_{133}, & C_1 &= C_2. \end{aligned} \tag{6.9}$$

These reduce the number of independent invariant integrals considerably. Note that not all of the 40 standard matrix elements of (5.14) appear in (6.6). This is due to the neglect of fermion masses and CP invariance of the box diagram.

This example shows that the reduction method is straightforward and universally applicable to one-loop Feynman diagrams, since they all have a similar structure.

6.2. Generic Feynman diagrams

The huge number of algebraic calculations makes the evaluation of each Feynman diagram very lengthy and tedious. Furthermore there are a large number of diagrams contributing to each process. Fortunately many of the diagrams resemble each other in their algebraic structure and can be considered as special cases of generic diagrams. These are the Feynman diagrams of a theory with only one generic scalar, fermion, vector boson and Faddeev-Popov ghost each and arbitrary renormalizable couplings between those fields (for more details see [34]). It suffices to do the algebraic calculations for these generic diagrams only. All actual diagrams are obtained from those by substituting the actual fields together with their coupling constants and masses. This saves a lot of work especially if there are many fields in the theory.

Clearly the generic diagrams can be calculated with the methods described above. The efficiency of generic diagrams is illustrated in the next section using the decay of the W -boson into massless fermions as example.

6.3. The decay $W \rightarrow f_i \bar{f}'_j$ for massless fermions

We will now apply the methods described above by calculating the one-loop amplitude for the decay of the W -boson into massless fermions

$$W^+(k) \rightarrow f_i(p_1) \bar{f}'_j(p_2). \tag{6.10}$$

In lowest order there is only one Feynman diagram (Fig. 6.2) leading to the amplitude

$$\mathcal{M}_0 = -\frac{eV_{ij}}{\sqrt{2}s_W} \bar{u}(p_1) \varepsilon(k) \omega_- v(p_2) = -\frac{eV_{ij}}{\sqrt{2}s_W} \mathcal{M}_1^-. \tag{6.11}$$

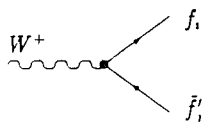


Fig. 6.2. Born diagram to $W \rightarrow f_i \bar{f}'_j$

Neglecting the fermion masses the amplitude (6.11) leads to the following lowest order decay width

$$\Gamma_0 = \frac{\alpha}{6} \frac{M_W}{2s_W^2} |V_{ij}|^2. \tag{6.12}$$

At one-loop order there are six loop diagrams and one counterterm diagram (Fig. 6.3; the counterterm is indicated by a cross).

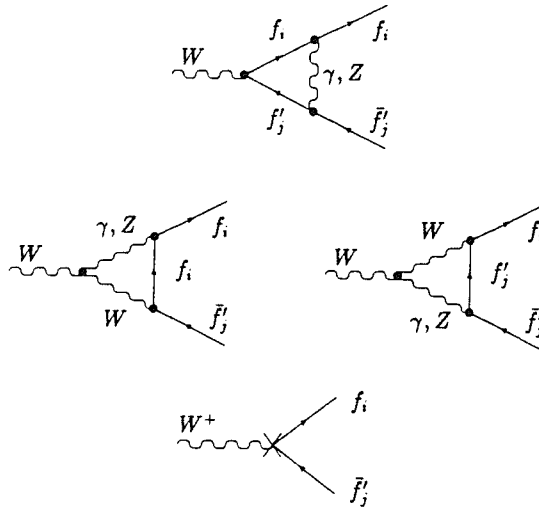


Fig. 6.3. One-loop diagrams and corresponding counterterm diagram to $W \rightarrow f_i \bar{f}_j$

The first two loop diagrams correspond to one generic diagram and the other four to another generic one. We first calculate the two generic diagrams. The expression for the first reads

$$\delta \mathcal{M}_1 = i \mu^{4-D} \int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 - M^2)(q + p_1)^2 (q - p_2)^2} \bar{u}(p_1) \gamma^\nu (g_1^- \omega_- + g_1^+ \omega_+) (\not{q} + \not{p}_1) \not{\epsilon} (g_3^- \omega_- + g_3^+ \omega_+) (\not{q} - \not{p}_2) \gamma_\nu (g_2^- \omega_- + g_2^+ \omega_+) v(p_2), \quad (6.13)$$

where g^\pm denote the generic left- and right-handed fermion-fermion-vector couplings. Simplification and decomposition into tensor integrals yields

$$\delta \mathcal{M}_1 = i \mu^{4-D} \int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 - M^2)(q + p_1)^2 (q - p_2)^2} \bar{u}(p_1) [-2(q - \not{p}_2) \not{\epsilon} (\not{q} + \not{p}_1) + (4 - D)(\not{q} \not{\epsilon} \not{q})] (g_1^- g_3^- g_2^- \omega_- + g_1^+ g_3^+ g_2^+ \omega_+) v(p_2) = -\frac{1}{16\pi^2} \bar{u}(p_1) [(2 - D) C_{\mu\nu} \gamma^\mu \not{\epsilon} \gamma^\nu + 2 C_\mu (\not{p}_2 \not{\epsilon} \gamma^\mu - \gamma^\mu \not{\epsilon} \not{p}_1) + 2 C_0 \not{p}_2 \not{\epsilon} \not{p}_1] (g_1^- g_2^- g_3^- \omega_- + g_1^+ g_2^+ g_3^+ \omega_+) v(p_2). \quad (6.14)$$

Insertion of the Lorentz decomposition and further simplification gives

$$\delta \mathcal{M}_1 = -\frac{1}{16\pi^2} (g_1^- g_3^- g_2^- \mathcal{M}_1^- + g_1^+ g_3^+ g_2^+ \mathcal{M}_1^+) [(2 - D)^2 C_{00} - 2k^2 (C_{12} + C_1 + C_2 + C_0)]. \quad (6.15)$$

Finally the reduction of the invariant integrals and the use of (4.55) leads to

$$\begin{aligned}\delta\mathcal{M}_1 &= -\frac{1}{16\pi^2}(g_1^- g_3^- g_2^- \mathcal{M}_1^- + g_1^+ g_3^+ g_2^+ \mathcal{M}_1^+) \\ &\quad [-2k^2 C_0(0, k^2, 0, M, 0, 0) (1 + \frac{M^2}{k^2})^2 \\ &\quad - B_0(k^2, 0, 0) (3 + 2\frac{M^2}{k^2}) + 2B_0(0, M, 0) (2 + \frac{M^2}{k^2}) - 2] \\ &= -\frac{1}{16\pi^2}(g_1^- g_3^- g_2^- \mathcal{M}_1^- + g_1^+ g_3^+ g_2^+ \mathcal{M}_1^+) \mathcal{V}_a(0, k^2, 0, M, 0, 0),\end{aligned}\quad (6.16)$$

where we introduced the generic vertex function \mathcal{V}_a which is defined in the general case in App. C.

Similarly we obtain for the second generic diagram

$$\begin{aligned}\delta\mathcal{M}_2 &= -i\mu^{4-D} \int \frac{d^D q}{(2\pi)^D} \frac{\bar{u}(p_1) \gamma^\nu (g_1^- \omega_- + g_1^+ \omega_+) (-\not{q}) \gamma_\nu (g_2^- \omega_- + g_2^+ \omega_+) v(p_2)}{q^2 [(q+p_1)^2 - M_1^2] [(q-p_2)^2 - M_2^2]} \\ &\quad g_3 [g_{\rho\mu}(p_1 + 2p_2 - q)_\nu - g_{\mu\nu}(2p_1 + p_2 + q)_\rho + g_{\nu\rho}(2q + p_1 - p_2)_\mu] \varepsilon^\mu \\ &= \frac{1}{16\pi^2} g_3 (g_1^- g_2^- \mathcal{M}_1^- + g_1^+ g_2^+ \mathcal{M}_1^+) [4(D-1)C_{00} - 2k^2(C_{12} + C_1 + C_2)] \\ &= \frac{1}{16\pi^2} g_3 (g_1^- g_2^- \mathcal{M}_1^- + g_1^+ g_2^+ \mathcal{M}_1^+) \\ &\quad [2(M_1^2 + M_2^2 + \frac{M_1^2 M_2^2}{k^2}) C_0(0, k^2, 0, 0, M_1, M_2) - (1 + \frac{M_1^2 + M_2^2}{k^2}) B_0(k^2, M_1, M_2) \\ &\quad + (2 + \frac{M_1^2}{k^2}) B_0(0, 0, M_1) + (2 + \frac{M_2^2}{k^2}) B_0(0, 0, M_2)] \\ &= \frac{1}{16\pi^2} g_3 (g_1^- g_2^- \mathcal{M}_1^- + g_1^+ g_2^+ \mathcal{M}_1^+) \mathcal{V}_b^-(0, k^2, 0, 0, M_1, M_2).\end{aligned}\quad (6.17)$$

The general definition of \mathcal{V}_b^- can again be found in App. C.

Inserting the actual couplings and masses of the six actual diagrams into the generic diagrams and adding the counterterm diagram, which can be easily obtained from the Feynman rules or the Born diagram, we find for the virtual one-loop corrections to the invariant amplitude for $W \rightarrow f_i \bar{f}_j$

$$\begin{aligned}\delta_s \mathcal{M} &= -\frac{e}{\sqrt{2}s_W} \frac{\alpha}{4\pi} V_{ij} \mathcal{M}_1^- \\ &\quad \{Q_f Q_f \mathcal{V}_a^-(0, M_W^2, 0, \lambda, 0, 0) + g_f^- g_f^- \mathcal{V}_a^-(0, M_W^2, 0, M_Z, 0, 0) \\ &\quad + Q_f \mathcal{V}_b^-(0, M_W^2, 0, 0, \lambda, M_W) - Q_f \mathcal{V}_b^-(0, M_W^2, 0, 0, M_W, \lambda) \\ &\quad + \frac{c_W}{s_W} g_f^- \mathcal{V}_b^-(0, M_W^2, 0, 0, M_Z, M_W) - \frac{c_W}{s_W} g_f^- \mathcal{V}_b^-(0, M_W^2, 0, 0, M_W, M_Z) \\ &\quad + \frac{1}{2} \delta Z_{ii}^{f, L+} + \frac{1}{2} \delta Z_{jj}^{f, L} + \frac{1}{2} \delta Z_W + \delta Z_e - \frac{\delta s_W}{s_W}\},\end{aligned}\quad (6.18)$$

The left- and right-handed couplings g_f^Z of the fermions to the Z -boson are defined in (A.14). Note that only one out of the four standard matrix elements (5.11) is contributing there and that we need no counterterm to the quark mixing matrix for massless fermions. The counterterms are expressed in terms of the self energies in Sect. 3.3. These have to be calculated to one-loop order to determine $\delta\mathcal{M}$ completely.

$\delta\mathcal{M}$ contains infrared divergencies. These are regularized with a photon mass λ . They drop out in the decay width if the contribution from the decay $W \rightarrow f_i \bar{f}_j \gamma$ is added. This will be discussed in more detail in Chap. 7.

The example above was rather simple. If we keep the fermion masses finite or consider processes with more external particles the number and complexity of Feynman diagrams raises considerably.

6.4. Computeralgebraic calculation of one-loop diagrams

The procedure of generation and algebraic reduction of Feynman diagrams as described above is algorithmic and can be implemented in symbolic computation systems. There are several attempts to create such systems for high energy physics calculations [35]. In addition there exist special packages written in general purpose languages [9, 10, 11, 12]. In particular the *MATHEMATICA* packages *FeynArts* [11] and *FeynCalc* [12] have been developed for the automatic calculation of one-loop diagrams following the approach outlined in this paper.

FeynArts generates all graphs to a given process in a specified model together with their combinatorial factors (weights). It yields both analytical expressions and drawings of the graphs. There is a version under development which uses the concept of generic diagrams. It creates all relevant generic graphs together with a list of all possible substitutions yielding the actual graphs.

FeynCalc performs the algebraic evaluation of Feynman diagrams. It starts from the output of *FeynArts* and uses exactly the reduction algorithm described above. It can deal with generic diagrams. The *FeynCalc* output can easily be translated into *FORTRAN* code.

7. Soft Photon Bremsstrahlung

As mentioned in the last chapter the virtual one-loop corrections to the decay matrix element $W \rightarrow f_i \bar{f}_j$ are infrared divergent. These divergencies originate from photonic corrections and show up in any process with charged external particles.

However, these processes are not of direct physical relevance since they cannot be distinguished experimentally from those involving additional soft external photons. Since the photons are massless their energies can be arbitrarily small and thus less than the resolution of any detector. Therefore in observable processes in addition to the basic process those with arbitrary numbers of soft photons are included.

For these observable processes one obtains theoretically satisfactory results. Adding incoherently the cross sections of all the different processes with arbitrary numbers of photons, all infrared divergencies cancel [36]. This cancellation takes place between the virtual photonic corrections and the real bremsstrahlung corrections order by order in perturbation theory. To one-loop order one only needs to consider single photon radiation. For the cancellations only the soft photons, i.e. photons with energy $k_0 \leq \Delta E$, are relevant, where ΔE is a cutoff parameter, which should be small compared to all

relevant energy scales. Photons with energies $k_0 > \Delta E$ are called hard. They can also yield sizable contributions especially arising from photon emission collinear to the external charged particles.

In Sect. 7.1 we introduce the soft photon approximation and show that in this approximation the bremsstrahlung diagrams are proportional to the Born diagrams. The corresponding soft photon cross section for arbitrary Born diagrams is given in Sect. 7.2.

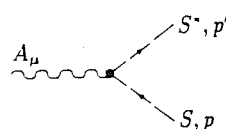
7.1. Soft photon approximation

Attaching soft photons to a charged external particle line of an arbitrary Feynman diagram yields diagrams which become singular for vanishing momentum of the soft photon. This divergence arises from the propagator of the charged particle generated by the inclusion of the radiated photon line. In the soft photon approximation the momenta of the radiated photons are neglected everywhere but in this singular propagator. This approximation is valid if the matrix element of the basic process does not change much if a photon with energy ΔE is emitted, i.e. the basic matrix element is a slowly varying function of the photon energy for $k_0 < \Delta E$. This is not the case if the basic process contains a narrow resonance as e.g. in $e^+e^- \rightarrow \mu^+\mu^-$. Then one must either choose ΔE small compared to the width of the resonance or take into account the strong variation exactly [37, 38].

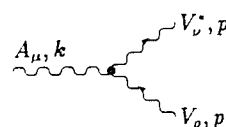
We now extract the soft photon matrix elements for external fermions, scalars and vector bosons. The general renormalizable couplings of these particles to the photon allowed by electromagnetic gauge invariance are (momenta are considered as incoming):



$$= -ieQ_F \gamma_\mu. \quad (7.1)$$



$$= -ieQ_S(p-p')_\mu. \quad (7.2)$$



$$= +ieQ_V g_{VQ}(p-p')_\mu - ie\kappa_V [k_\theta g_{\mu\nu} - k_\nu g_{\mu\theta}]. \quad (7.3)$$

Quartic boson couplings do not give rise to IR-singularities and are thus not relevant in the soft photon approximation. The terms involving the charges Q are obtained directly from the covariant derivative with respect to QED. The term proportional to κ_V , which contributes only to the magnetic moment, is gauge invariant by itself. Further terms present in the γWW coupling in the SM do not contribute for physical vector bosons. Since we will use the unitary gauge in this section they drop out. In renormalizable gauges their contributions are cancelled by those of the corresponding unphysical Higgs bosons.

Consider first radiation from a fermion line. Let the basic matrix element without soft photons be

$$\mathcal{M}_0 = A(p) u(p) = \xrightarrow{F} \text{[shaded circle]}, \quad (7.4)$$

where $u(p)$ is the fermion spinor with momentum p , $p^2 = m^2$ and $A(p)$ the remaining part of the matrix element. Inserting one photon (polarization vector ε , momentum k) into the fermion line yields

$$\begin{aligned} \mathcal{M}_1 &= \xrightarrow{F, p} \text{[shaded circle]} \\ &= A(p-k) \frac{i(\not{p} - \not{k} + m)}{(p-k)^2 - m^2} (-ie Q_F \not{\varepsilon}) u(p). \end{aligned} \quad (7.5)$$

Anticommuting $\not{p} - \frac{1}{2}\not{k}$ with $\not{\varepsilon}$ and using the Dirac equation this can be written as

$$\mathcal{M}_1 = \frac{e Q_F}{-2pk} A(p-k) [2\varepsilon p - i\varepsilon^\mu \sigma_{\mu\nu} k^\nu] u(p), \quad (7.6)$$

where $\sigma_{\mu\nu} = \frac{i}{2} [\gamma_\mu, \gamma_\nu]$. The denominator $\frac{1}{2pk}$ contains the IR-singularity. Neglecting all terms proportional k in the numerator we obtain the soft photon approximation

$$\mathcal{M}_{1,s} = -e Q_F \frac{\varepsilon p}{kp} A(p) u(p) = -e Q_F \frac{\varepsilon p}{kp} \mathcal{M}_0. \quad (7.7)$$

Note that the contributions of the magnetic moment term, the second term in the square bracket in (7.6), are neglected in the soft photon approximation and that the soft photon matrix element is proportional to the Born matrix element. For an outgoing fermion ($\bar{u}(p)$) one finds in the same way

$$\mathcal{M}_{1,s} = e Q_F \frac{\varepsilon p}{kp} \mathcal{M}_0. \quad (7.8)$$

This is equivalent to (7.7) apart from a minus sign originating from the different charge flow.

For an external vector line with polarization vector $\varepsilon_V(p)$ the basic matrix element is

$$\mathcal{M}_0 = A_\sigma(p) \varepsilon_V^\sigma(p) = \text{[wavy line]} \text{[shaded circle]}. \quad (7.9)$$

The corresponding soft photon contribution is obtained from

$$\mathcal{M}_1 = A_\sigma(p-k) \frac{-i}{(p-k)^2 - M^2} \left(g^{\sigma\nu} - \frac{(p-k)^\sigma (p-k)^\nu}{M^2} \right) \varepsilon_V^\sigma(p) \varepsilon^\mu(k) i e [Q_V g_{\nu\sigma} (2p-k)_\mu - \kappa_V (k_\sigma g_{\mu\nu} - k_\nu g_{\mu\sigma})], \tag{7.10}$$

using

$$\varepsilon_V \cdot p = 0 \tag{7.11}$$

as

$$\mathcal{M}_{1,s} = -e Q_V \frac{p^\varepsilon}{pk} \mathcal{M}_0. \tag{7.12}$$

It is proportional to the Born matrix element and independent of the contribution of the magnetic moment κ_V of the boson V . Again an outgoing vector yields an extra minus sign.

The soft photon matrix element for an external scalar line is derived analogously.

Radiation from internal charged lines or quartic vertices does not lead to IR-singularities and is neglected in the soft photon approximation.

Summarizing, the $O(\alpha)$ soft photon matrix element corresponding to an arbitrary matrix element \mathcal{M}_0 can be written as

$$\mathcal{M}_{1,s} = -e \mathcal{M}_0 \sum_i (\pm Q_i) \frac{\varepsilon p_i}{k p_i}, \tag{7.13}$$

where p_i, Q_i are the momentum and the charge of the i -th external particle and k is the outgoing photon momentum. The \pm sign refers to charges flowing into or out of the diagram, respectively. The soft photon matrix element is always proportional to the Born matrix element. The proportionality factor depends only on the charges and momenta of the external particles.

7.2. Soft photon cross section

The soft photon cross section is obtained by squaring the soft photon matrix element (7.13), summing over the photon polarizations and integrating over the photon phase space with $|\mathbf{k}| \leq \Delta E$

$$\left(\frac{d\sigma}{d\Omega} \right)_s = - \left(\frac{d\sigma}{d\Omega} \right)_0 \frac{e^2}{(2\pi)^3} \int_{|\mathbf{k}| \leq \Delta E} \frac{d^3k}{2\omega_k} \sum_{ij} \frac{\pm p_i p_j Q_i Q_j}{p_i k p_j k}, \tag{7.14}$$

where

$$\omega_k = \sqrt{\mathbf{k}^2 + \lambda^2} \tag{7.15}$$

and \pm refers to the relative sign of the i -th and j -th term in (7.13). As in the virtual corrections the IR-singularities are regularized by the photon mass λ . Note that these integrals are not Lorentz-invariant due to the integration region. The basic integrals

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have been worked out e.g. by 'T HOOFT and VELTMAN [28].

The result is

$$I_{ij} = 4\pi \frac{\alpha p_i p_j}{(\alpha p_i)^2 - p_j^2} \left\{ \frac{1}{2} \log \frac{(\alpha p_i)^2}{p_j^2} \log \frac{4\Delta E^2}{\lambda^2} + \left[\frac{1}{4} \log^2 \frac{u_0 - |\mathbf{u}|}{u_0 + |\mathbf{u}|} + \text{Li}_2 \left(1 - \frac{u_0 + |\mathbf{u}|}{v} \right) + \text{Li}_2 \left(1 - \frac{u_0 - |\mathbf{u}|}{v} \right) \right]_{u=p_j}^{u=\alpha p_i} \right\}, \quad (7.17)$$

with

$$v = \frac{(\alpha p_i)^2 - p_j^2}{2(\alpha p_{i0} - p_{j0})}, \quad (7.18)$$

and α defined through

$$\alpha^2 p_i^2 - 2\alpha p_i p_j + p_j^2 = 0, \quad \frac{\alpha p_{i0} - p_{j0}}{p_{j0}} > 0. \quad (7.19)$$

For $p_i = p_j$ this simplifies to

$$I_{ii} = 2\pi \left\{ \log \frac{4\Delta E^2}{\lambda^2} + \frac{p_0}{|\mathbf{p}|} \log \frac{p_0 - |\mathbf{p}|}{p_0 + |\mathbf{p}|} \right\}, \quad (7.20)$$

and for $\mathbf{p}_i = -\mathbf{p}_j = \mathbf{p}$

$$I_{ij} = 2\pi \frac{pq}{(p_0 + q_0)|\mathbf{p}|} \left\{ \frac{1}{2} \log \frac{p_0 + |\mathbf{p}|}{p_0 - |\mathbf{p}|} \log \frac{4\Delta E^2}{\lambda^2} - \text{Li}_2 \left(\frac{2|\mathbf{p}|}{p_0 + |\mathbf{p}|} \right) - \frac{1}{4} \log^2 \frac{p_0 + |\mathbf{p}|}{p_0 - |\mathbf{p}|} + \frac{1}{2} \log \frac{q_0 + |\mathbf{p}|}{q_0 - |\mathbf{p}|} \log \frac{4\Delta E^2}{\lambda^2} - \text{Li}_2 \left(\frac{2|\mathbf{p}|}{q_0 + |\mathbf{p}|} \right) - \frac{1}{4} \log^2 \frac{q_0 + |\mathbf{p}|}{q_0 - |\mathbf{p}|} \right\}. \quad (7.21)$$

Inserting the results for I_{ij} into (7.14) yields the soft photon cross section. Adding it to the one-loop corrected cross section for the corresponding basic process the IR-divergencies cancel and the limit $\lambda \rightarrow 0$ can be taken.

Although the inclusion of the real soft photon emission is sufficient to obtain IR-finite results, it is often not adequate for real experiments, because realistic detectors do not provide a sufficiently small resolution $\Delta E/E$ necessary for the validity of the soft photon approximation. Therefore also hard photons (with $k_0 > \Delta E$) are important. Their contribution is UV- and IR-finite and can be treated separately. One merely has to make sure that the soft and hard part are properly adapted to each other.

Hard photon corrections are treated with methods different from the ones presented in this work. Their contribution depends sensitively on the experimental setup. They are usually incorporated by Monte Carlo simulations [15].

8. Input parameters and leading higher order contributions

In order to complete all ingredients necessary for the calculation of radiative corrections we have to specify the input parameters. This is done in Sect. 8.1. The leading higher order corrections which become important for precision experiments are discussed in Sect. 8.2.

8.1. Input parameters

In its original symmetric version the SM depends on the parameters (2.21), which are essentially the couplings allowed by the $SU(2)_W \times U(1)_Y$ symmetry. These were replaced by the physical parameters (2.22), i.e. the particle masses, the electromagnetic coupling constant, and the quark mixing matrix. In the on-shell renormalization scheme the renormalized parameters are equal to these physical parameters in all orders of perturbation theory.

The numerical values of the physical parameters must be fixed through experimental input. However, this input may not necessarily consist of direct measurements of the renormalized parameters; it may be obtained from any suitable set of experimental results. In practice one uses those experiments which have the highest experimental accuracy and theoretical reliability. This criterion is certainly fulfilled for the following set of parameters whose numerical values are taken from [39]:

- the fine structure constant

$$\alpha = 1/137.0359895(61)$$

corresponding to the classical electron charge $e = \sqrt{4\pi\alpha}$.

- the masses of the charged leptons

$$m_e = 0.51099906(15) \text{ MeV}, \quad m_\mu = 105.658387(34) \text{ MeV},$$

$$m_\tau = 1784.1_{-3.6}^{+2.7} \text{ MeV}.$$

- the mass of the Z-boson [5]

$$M_Z = 91.177(21) \text{ GeV},$$

- and the Fermi constant

$$G_F = 1.16637(2) \cdot 10^{-5} \text{ GeV}^{-2},$$

which is directly related to the muon lifetime.

We do not use the W-mass as input parameter because it is experimentally not known with comparable accuracy.

Besides the above listed well known parameters the still unknown masses of the top quark and the Higgs scalar are kept as free parameters. If the minimal SM is correct, the present experimental data restrict the top quark mass to the region $80 \text{ GeV} < m_t < 200 \text{ GeV}$ [5, 8]. For the Higgs mass we use $40 \text{ GeV} < M_H < 1 \text{ TeV}$, where the lower bound is experimental [5] and the upper bound is favored by theoretical consistency arguments. If not stated otherwise we will use the values $m_t = 140 \text{ GeV}$ and $M_H = 100 \text{ GeV}$.

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The quark mixing matrix elements V_{ij} are directly taken from experiment. We use the parametrization of Harari and Leurer [40] as advocated by the Particle Data Group and choose the following numerical values for the parameters in agreement with [39]

$$s_{12} = 0.220, \quad s_{23} = 0.046, \quad s_{13} = 0.007 \quad (8.1)$$

and $\delta = 0$ for simplicity. This yields approximately the following numbers for the quark mixing matrix elements:

$$\begin{aligned} V_{ud} &= 0.975, & V_{us} &= 0.220, & V_{ub} &= 0.007, \\ V_{cd} &= -0.220, & V_{cs} &= 0.974, & V_{cb} &= 0.046, \\ V_{td} &= 0.003, & V_{ts} &= -0.046, & V_{tb} &= 0.999. \end{aligned} \quad (8.2)$$

It remains to discuss the masses m_q of the light quarks ($q = d, u, s, c, b$). In the electro-weak Lagrangian the quarks are treated as free particles with appropriate masses. This is not correct due to the presence of the strong interaction. Therefore the quark masses can at best be considered as somewhat effective parameters. Fortunately in typical high energy experiments ($s \gg m_q^2$) theoretical predictions depend on the quark masses only through universal quantities such as the hadronic vacuum polarization or the quark structure functions. These can be directly determined from experiment. Nonuniversal contributions are suppressed as m_q^2/s and thus negligible for sufficiently high energies.

For processes without external quarks only the hadronic contribution to the vacuum polarization

$$\Pi^{AA}(s) = \frac{\Sigma_T^{AA}(s)}{s} \quad (8.3)$$

is relevant. In perturbation theory the contribution of light quarks is given by

$$\hat{\Pi}_{\text{had}}^{AA}(s) = 3 \frac{\alpha}{3\pi} \sum_{d,u,s,c,b} Q_q^2 \left(\frac{5}{3} - \log \frac{-s - i\epsilon}{m_q^2} \right). \quad (8.4)$$

The large logarithmic terms contained in (8.4) constitute a dominant contribution to the radiative corrections. They originate from the charge renormalization in the on-shell scheme at zero momentum transfer (see eq. 3.32) involving

$$\Pi^{AA}(0) = \left. \frac{\partial \Sigma_T^{AA}(k^2)}{\partial k^2} \right|_{k^2=0}. \quad (8.5)$$

In this quantity nonperturbative strong interaction effects cannot be neglected. Since no reliable theoretical predictions are available one has to extract $\Pi_{\text{had}}^{AA}(0)$ from experimental data. Writing

$$\begin{aligned} \Pi_{\text{had}}^{AA}(0) &= \Pi_{\text{had}}^{AA}(0) - \text{Re} \Pi_{\text{had}}^{AA}(s) + \text{Re} \Pi_{\text{had}}^{AA}(s) \\ &= -\text{Re} \hat{\Pi}_{\text{had}}^{AA}(s) + \text{Re} \Pi_{\text{had}}^{AA}(s), \end{aligned} \quad (8.6)$$

the unrenormalized hadronic vacuum polarization $\Pi_{\text{had}}^{AA}(s)$ can be evaluated perturbatively for $s \gg m_q^2$ and the renormalized one $\text{Re} \hat{\Pi}_{\text{had}}^{AA}(s)$ is given by the dispersion relation