Name: SOLUTIONS

PHY 5246: Theoretical Dynamics, Fall 2015

September 28^{th}, 2015
Midterm Exam # 1

Always remember to write full work for what you do. This will help your grade in case of incomplete or wrong answers. Also, no credit will be given for an answer, even if correct, if you give no justification for it.

Write your final answers on the sheets provided. You may separate them as long as you put your name on each of them. We will staple them when you hand them in. Ask if you need extra sheets, they will be provided. Remember to put your name on each of them and add them after the problem they refer to.

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Problem 1

A particle of mass m is constrained to move on the surface of a cylinder defined by $x^2+y^2=R^2$. The particle is subject to a force directed toward the origin and proportional to the the distance of the particle from the origin: $\mathbf{F}=-k\mathbf{r}$. There is no gravitational force acting on m.

(1.a) Write the Lagrangian of the system using cylindrical coordinates. Can you tell if the system admits one or more conserved quantities (or *first integrals*)?

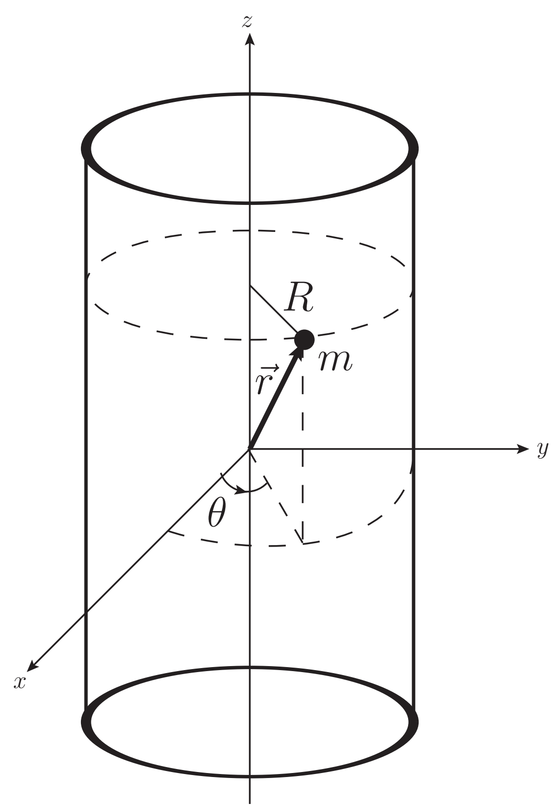
(1.b) Find the equations of motion using the Euler-Lagrange method, integrate them, and tell how the bead moves.

(1.c) Find the force of constraint acting on the bead. If you prefer you can solve (**1.b**) and (**1.c**) together.

(1.a)

Using cylindrical coordinates (ρ,θ,z), with $\hat{\rho}=x\hat{\mathbf{x}}+y\hat{\mathbf{y}}$, the constraint of the problem is simply implemented by fixing the radial coordinate to $\rho=R$, and using $\{\theta,z\}$ as generalized coordinates, such that:

$$\begin{cases} x &= R\cos\theta \\ y &= R\sin\theta \\ z &= z \end{cases}$$



Notice that the force is $\mathbf{F}=-k\mathbf{r}$, where $\mathbf{r}=r\hat{\mathbf{r}}$ is the position vector of the bead and r is the radial coordinate of spherical coordinates, s.t. $r^2=\rho^2+z^2=R^2+z^2$. We can find the corresponding potential energy using directly spherical coordinates (simplest), in which case:

$$\mathbf{F}=-k\mathbf{r}=-\boldsymbol{\nabla}V(r)=-\frac{\partial V(r)}{\partial r}\hat{\mathbf{r}}$$

such that,

$$\frac{\partial V(r)}{\partial r}=-kr\Rightarrow V(r)=\frac{1}{2}kr^2\ .$$

We could also rewrite **F** in cylindrical coordinates:

$$\mathbf{F} = -k\rho\hat{\boldsymbol{\rho}} - kz\hat{\mathbf{z}} = -\boldsymbol{\nabla} V(\rho,\theta,z) = -\frac{\partial V}{\partial \rho}\hat{\boldsymbol{\rho}} - \frac{\partial V}{\partial z}\hat{\mathbf{z}} \ ,$$

such that

$$V(\rho,\theta,z) = \frac{1}{2}k(\rho^2+z^2) = \frac{1}{2}r^2 \ ,$$

in agreement with our previous result. The kinetic energy can then be expressed in cylindrical coordinates as,

$$\begin{aligned} T &= \frac{m}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2) = \frac{1}{2}m(R^2\dot{\theta}^2+\dot{z}^2) \\ V &= \frac{1}{2}kr^2 = \frac{1}{2}k(R^2+z^2) \\ &= \frac{1}{2}kz^2 + \text{const.} \end{aligned}$$

such that, the Lagrangian is

$$L = T - V = \frac{1}{2}m(R^2\dot{\theta}^2+\dot{z}^2) - \frac{1}{2}kz^2 \ .$$

We notice that since L does not depend on θ we have a conserved quantity, the angular momentum in the $\hat{\theta}$ direction,

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \text{constant} \ .$$

Moreover, since the only applied force is conservative, energy must be conserved. You can proof this directly, by taking the derivative of E with respect to t and using the equations of motion that you will derive in **(1.b)**, or you can observe that: 1) the energy function h is equal to the energy E, since the force is conservative, and the constraint is time-independent, and 2) h is conserved because the Lagrangian does not depend *explicitly* on time, i.e. $\partial L/dt = 0$ (remember that $dh/dt = -\partial L/dt$).

(1.b)

The equations of motion are given by

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 \Rightarrow p_\theta = mR^2\dot{\theta} = \text{const.} \Rightarrow \dot{\theta} = \frac{p_\theta}{mR^2} = \text{const.} \tag{1}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0 \Rightarrow m\ddot{z} + kz = 0 \Rightarrow \ddot{z} + \omega^2 z = 0. \tag{2}$$

So, the particle rotates about the z axis with constant angular velocity $\dot{\theta}$ and oscillates in the $\hat{\mathbf{z}}$ direction as a simple harmonic oscillator with frequency $\omega = \sqrt{k/m}$ (about a position that it is determined by some initial conditions, not given in the problem).

Having the equations of motion we can also explicitly check that the energy E is conserved (this was left on hold from point **(1.a)**). The derivative of E with respect to time is:

$$\begin{aligned}
\frac{dE}{dt} &= \frac{d}{dt}(T+V) = \frac{d}{dt}\left(\frac{1}{2}m(R^2\dot{\theta}^2 + \frac{1}{2}kz^2)\right) \\
&= \cancel{2}\frac{1}{\cancel{2}}mR^2\dot{\theta}\ddot{\theta} + \cancel{2}\frac{1}{\cancel{2}}m\dot{z}\ddot{z} + \cancel{2}\frac{1}{\cancel{2}}kz\dot{z} \\
&= \dot{z}(m\ddot{z} + kz) = 0 \ ,
\end{aligned}$$

where, in the last two lines, we have used the equations of motion (1) and (2).

(1.c)

The constraint is in this case a geometrical constraint: the bead has to move on a cylindrical surface of fixed radius $R = a$, for a some arbitrary constant. The differential form of the equation of the constraint is then,

$$a_R dR = 0 \ \text{ with } \ a_R = 1 \ .$$

The force of the constraint will be at all times orthogonal to the surface, so we expect it to have only one component, in the radial cylindrical direction. In order to find its components, we need to introduce a number of Lagrange multipliers λ_j equal to the number of equation of constraints (one in our case), consider all three coordinates as generalized coordinates, rewrite the Lagrangian as,

$$L = T - V = \frac{1}{2}m(\dot{R}^2 + R^2\dot{\theta}^2 + \dot{z}^2) - \frac{1}{2}k(R^2 + z^2) \ ,$$

and write the Euler-Lagrange equations of motion in the generalized form,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = a_j\lambda_j \ .$$

In our case only the R equation of motion will be modified, and will take the form,

$$m\ddot{R} - mR\dot{\theta}^2 + kR = \lambda \ . \tag{3}$$

Solving Eq. (3) together with Eqs. (1) and (2), plus the constraint condition: $R = a$ and $\dot{R} = \ddot{R} = 0$, we get,

$$\lambda = ka - ma\dot{\theta}^2 = ka - \frac{l^2}{ma^3} \ ,$$

which is the force of the constraint (just one component, in the radial cylindrical direction, as expected).

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Problem 2

Consider a simple plane pendulum consisting of a mass m attached to a string of length l. After the pendulum is set in motion, the length of the string is shortened at a constant rate

$$\frac{dl}{dt} = -\alpha = \text{constant} \ .$$

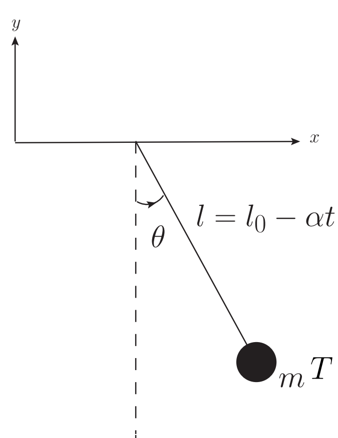
The suspension point remains fixed.

(2.a) Compute the Lagrangian function and find the equation of motion. Show that the equation you found corresponds to the $\hat{\theta}$ component of $\mathbf{F} = m\mathbf{a}$ applied to the pendulum.

(2.b) Write the *energy function* (h) of the system. Is it equal to the energy of the system and why?

(2.c) Is the energy function h an integral of motion? Is the energy conserved? Explain your results.

(2.a)



The coordinates for this problem are

$$\begin{cases} x &= l\sin\theta \\ y &= -l\cos\theta \end{cases}$$

with $l = l_0 - \alpha t$ so that $\dot{l} = -\alpha$. The only generalized coordinate is $\{\theta\}$ (l is a time dependent constraint, it tells us how long the string is; the dynamics of how this happen is neither explained nor given, so you should not consider it as a coordinate!). The Lagrangian is given by finding the kinetic and potential energies

$$\begin{aligned} T &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m(\dot{l}^2 + l^2\dot{\theta}^2) \\ &= \frac{1}{2}m(\alpha^2 + l^2\dot{\theta}^2) \\ V &= -mgl\cos\theta \\ L &= T - V = \frac{1}{2}m(\alpha^2 + l^2\dot{\theta}^2) + mgl\cos\theta. \end{aligned}$$

The equation of motion is

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = 0 \Rightarrow m[l^2\ddot{\theta} + 2l\dot{l}\dot{\theta}] + mgl\sin\theta = 0$$

$$\Rightarrow \ddot{\theta} - \frac{2\alpha}{l}\dot{\theta} + \frac{g}{l}\sin\theta = 0. \tag{4}$$

which can be simply interpreted as the θ component of 2^{nd} Newton's law for this system, i.e.

$$ma_\theta = m(l\ddot{\theta} + 2\dot{l}\dot{\theta}) = -mg\sin\theta = F_\theta \ . \tag{5}$$

(2.b)

The energy function h is

$$\begin{aligned} h &= \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L = \dot{\theta} m l^2 \dot{\theta} - \frac{1}{2}m(\alpha^2 + l^2\dot{\theta}^2) - mgl\cos\theta \\ &= -\frac{1}{2}m\alpha^2 + \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta \ , \end{aligned}$$

while the energy is given by

$$\begin{aligned} E &= T + V = \frac{1}{2}m(\alpha^2 + l^2\dot{\theta}^2) - mgl\cos\theta \\ &= h + m\alpha^2. \end{aligned}$$

This shows that the two quantities are not equal, they differ by a constant term $h = E - m\alpha^2$. This is to be expected since the relation between Cartesian and generalized coordinates is time dependent.

(2.c)

Since $l = l(t)$,

$$\frac{dh}{dt} = \frac{dE}{dt} = -\frac{\partial L}{\partial t} \neq 0.$$

and h is not is not an integral of motion. The total energy E is not conserved either. Actually, since $h = E + \text{const.}$, $dE/dt = dh/dt \neq 0$. The physical interpretation is that the system is not closed; the mechanical energy that we are adding by shortening the string is not being accounted for.

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Problem 3

A particle of mass m and charge e moves with velocity $\mathbf{v}$ in the presence of an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$, and is therefore subject to a force

$$\mathbf{F} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \ .$$

(3.a) Show that the particle's equations of motion can be derived from the Lagrangian

$$L = \frac{1}{2}mv^2 - e\phi + e\mathbf{A} \cdot \mathbf{v} \ ,$$

where ϕ and $\mathbf{A}$ are the scalar and vector potentials in terms of which the fields $\mathbf{E}$ and $\mathbf{B}$ can be expressed as

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \ , \quad \mathbf{B} = \nabla \times \mathbf{A} \ .$$

(3.b) Show that the *energy function* of the system is

$$H = \frac{1}{2}mv^2 + e\phi \ .$$

Assuming that $\mathbf{A}$ and ϕ are time independent, show that H is conserved. Is H equal to the energy of the system? Explain why.

(3.c) Show that the equations of motion are invariant under a *gauge transformation* of the form

$$\begin{aligned} A_i' &= A_i + \partial_i \Lambda \ , \\ \phi' &= \phi - \partial_t \Lambda \ , \end{aligned}$$

where $\Lambda = \Lambda(\vec{x}, t)$ is a generic scalar function of $\vec{x}$ and t, $\partial_i = \partial/\partial x_i$, and $\partial_t = \partial/\partial t$ (*Hint*: consider how the transformation acts on the Lagrangian).

(3.a)

Starting from the expression of the force, one can work out $\mathbf{F} = m\mathbf{a}$ component by component (i.e. $m\ddot{x} = eE_x + e(\mathbf{x} \times \mathbf{B})_x = \ldots$), or, using a more compact notation, one can write that the i^{th} component of the force is

$$F_i = e(E_i + \epsilon_{ijk} v_j B_k) \ , \tag{6}$$

where,

$$E_i = -\partial_i \phi - \partial_t A_i \ , \qquad (7)$$
$$B_i = \epsilon_{ijk} \partial_j A_k \ ,$$

and summation over repeated indeces is always understood. The equations of motion then are

$$\begin{aligned} m v_i &= -e\partial_i\phi - e\partial_t A_i + e\epsilon_{ijk} v_j \epsilon_{klm} \partial_l A_m \qquad (8) \\ &= -e\partial_i\phi - e\partial_t A_i + e v_j \partial_l A_m (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \\ &- -e\partial_i\phi - e\partial_t A_i + e(v_m \partial_i A_m - v_l \partial_l A_i) \\ &= -e\partial_i\phi + e v_m \partial_i A_m - e\frac{dA_i}{dt} \\ &= -e\partial_i(\phi - v_m A_m) - e\partial_i v_m A_m - e\frac{dA_i}{dt} \end{aligned}$$

and are equivalent to (notice that $\partial_i v_m = 0$),

$$\frac{d}{dt}(m v_i + e A_i) + \partial_i (e\phi - e v_j A_j) = \frac{d}{dt}\frac{\partial L}{\partial v_i} - \frac{\partial L}{\partial x_i} \ , \qquad (9)$$

where L is the Lagrangian function defined as,

$$L = \frac{1}{2} m v^2 - e\phi + e\mathbf{A}\cdot\mathbf{v} \ . \qquad (10)$$

(3.b)

The *energy function* is defined as

$$\begin{aligned} H &= \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L \qquad (11) \\ &= \frac{1}{2} m v^2 + e\mathbf{v}\cdot\mathbf{A} + e\phi - e\mathbf{v}\cdot\mathbf{A} = \frac{1}{2} m v^2 + e\phi \ . \end{aligned}$$

H is conserved since (under the assumption that ϕ and $\mathbf{A}$ are time independent)

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} = 0 \ . \qquad (12)$$

This is expected, since the magnetic part of the force ($\mathbf{v} \times \mathbf{B}$) does not do work on the charge (being orthogonal to $\mathbf{v}$). However, H is not equal to the energy of the system, it is just the sum of the kinetic energy and the scalar electromagnetic potential.

(3.c)

Under the given *gauge transformation* the Lagrangian L transforms as follows

$$L \to L' = L + e\partial_t \Lambda + e\mathbf{v} \cdot \nabla \Lambda = L + e\frac{d\Lambda}{dt} \quad , \tag{13}$$

Since L and L' differ only by a total derivative with respect to time, they yield exactly the same equations of motion. This is expressed by saying that the Lagrangian is *gauge invariant*.