Name:Solutions

# PHY 5246: Theoretical Dynamics, Fall 2015

November $2^{nd}$, 2015
Midterm Exam # 2

Always remember to write full work for what you do. This will help your grade in case of incomplete or wrong answers. Also, no credit will be given for an answer, even if correct, if you give no justification for it.

Write your final answers on the sheets provided. You may separate them as long as you put your name on each of them. We will staple them when you hand them in. Ask if you need extra sheets, they will be provided. Remember to put your name on each of them and add them after the problem they refer to.

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## Problem 1

A particle of mass $m$ moves in an attractive central-force field

$$\mathbf{F}(r) = -\frac{k}{r^{\beta+1}}\hat{\mathbf{r}} \ ,$$

for $k$ and $\beta$ (positive) constants, and $\hat{\mathbf{r}}$ the unit vector in the radial direction from the center of force.

**(1.a)** Write the Lagrangian of the particle $m$ and explain your choice of generalized coordinates (how many generalized coordinates? why?)

For a central-force motion the vector angular momentum ($\mathbf{l}$) is conserved since $\mathbf{N} = \mathbf{r} \times \mathbf{F}(r) = 0$ and $\mathbf{N} = \dot{\mathbf{l}}$. This implies that the motion is planar, since the vectors $\mathbf{r}$ and $\dot{\mathbf{r}}$ are constrained to always lye in a plane orthogonal to $\mathbf{l}$, and $\mathbf{l}$ is constant. It is then natural to choose planar polar coordinates ($r$ and $\theta$) to describe the motion, and write the Lagrangian as,

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{\beta}\frac{k}{r^\beta} \ , \tag{1}$$

where we have derived the expression of $V(r) = -k/\beta/r^\beta$ from $F(r) = -\partial V/\partial r$.

**(1.b)** Construct the effective potential $V_{\text{eff}}(r)$ for which the radial equation of motion reads:

$$m\ddot{r} = -\frac{\partial V_{\text{eff}}(r)}{\partial r} \ .$$

The equation of motion for $\theta$ expresses the conservation of the magnitude of the angular momentum, i.e.

$$\frac{d}{dr}(mr^2\dot{\theta}) = 0 \ \rightarrow \ mr^2\dot{\theta} = l_0 \ , \tag{2}$$

where $l_0$ is a constant (of motion). On the other hand, the equation of motion for $r$ reads,

$$m\ddot{r} - mr\dot{\theta}^2 - \frac{k}{\beta}\frac{(-\beta)}{r^{\beta+1}} = 0 \ , \tag{3}$$

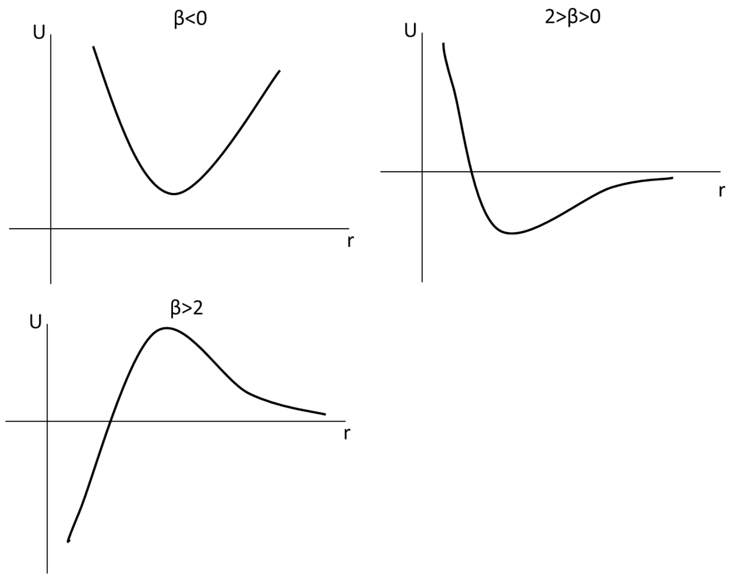
or, using Eq. (2),

$$m\ddot{r} - \frac{l_0^2}{mr^3} + \frac{k}{r^{\beta+1}} = 0 \ \rightarrow \ m\ddot{r} = \frac{l_0^2}{mr^3} - \frac{k}{r^{\beta+1}} = F_{\text{eff}}(r) = -\frac{\partial V_{\text{eff}}(r)}{\partial r} \ , \tag{4}$$

where we have rewritten it as the equation of motion for a one-dimensional system subject to an *effective* central force $F_{\text{eff}}(r)$ which can be defined in terms of an *effective* potential

$$V_{\text{eff}}(r) = -\int dr F_{\text{eff}}(r) = \frac{l_0^2}{2mr^2} - \frac{1}{\beta}\frac{k}{r^\beta} = \frac{l_0^2}{2mr^2} + V(r) \; . \tag{5}$$

**(1.c)** Sketch $V_{\text{eff}}(r)$ for the case **i)** $\beta < 0$, **ii)** $0 < \beta < 2$, and **iii)** $\beta > 2$. For what values of $\beta$ does a stable circular orbit exist? For what values of $\beta$ are all orbits bounded?




One can see from the sketches in the figure that a stable circular orbit exists only for $\beta < 0$ and $0 < \beta < 2$, since the effective potential has a minimum for $r = r_0$, where $r_0$ defines the radius of the circular orbit. For $\beta > 2$ $V_{\text{eff}}(r)$ has a maximum for $r = r_0$ and the circular orbit is not stable. One also sees that only for $\beta < 0$ all the orbits are bounded.

**(1.d)** For those values of $\beta$ which support the existence of a stable circular orbit, calculate the radius, $r_0$, of the circular orbit in terms of the (conserved) angular momentum ($l_0$) and other constants.

The radius of the circular stable orbit is obtained from the equation

$$\frac{\partial V_{\text{eff}}(r)}{\partial r} = -\frac{l_0^2}{mr^3} + \frac{1}{\beta}\beta\frac{k}{r^{\beta+1}} = 0 \;\rightarrow\; \frac{1}{r^3}\left(-\frac{l_0^2}{m} + \frac{k}{r^{\beta-2}}\right) = 0 \; , \tag{6}$$

from which one can derive $r_0$ as

$$\frac{l_0^2}{m} = \frac{k}{r_0^{\beta-2}} \;\rightarrow\; r_0^{\beta-2} = \frac{mk}{l_0^2} \;\rightarrow\; r_0 = \left(\frac{l_0^2}{mk}\right)^{\frac{1}{2-\beta}} \; . \tag{7}$$

**(1.e)** Let $r = r_0 + \eta$ and derive the equation of motion for radial deviations, assuming $\eta$ small. Under what conditions will the perturbed orbit be closed?

The equation of motion for $r = r_0 + \eta$ is

$$m\ddot{\eta} = \frac{l_0^2}{m(r_0+\eta)^3} - \frac{k}{(r_0+\eta)^{\beta+1}} \ . \tag{8}$$

Expanding in $\eta$ one gets,

$$m\ddot{\eta} = \frac{l_0^2}{mr_0^3}\left(1 - 3\frac{\eta}{r_0}\right) - \frac{k}{r_0^{\beta+1}}\left(1 - (\beta+1)\frac{\eta}{r_0}\right) + O(\eta^2) \ , \tag{9}$$

which can be recast in the form,

$$m\ddot{\eta} = \left(\frac{l_0^2}{mr_0^3} - \frac{k}{r_0^{\beta+1}}\right) - \eta\left(\frac{3l_0^2}{mr_0^4} - \frac{k(\beta+1)}{r_0^{\beta+2}}\right) + O(\eta^2) \ , \tag{10}$$

and finally, by observing that the first parenthesis vanishes due to the condition on $V_{\text{eff}}$ that defines $r_0$, one gets,

$$\ddot{\eta} + \omega_r^2 \eta = 0 \ , \tag{11}$$

with

$$\omega_r^2 = \frac{l_0^2}{m^2 r_0^4}(2-\beta) \ \ \rightarrow \ \ \omega_r = \sqrt{2-\beta}\frac{l_0}{mr_0^2} \ , \tag{12}$$

which shows how the radial displacement undergoes harmonic oscillations of frequency $\omega_r$. When the orbit is closed the frequency of radial oscillations has to be commensurate with the angular frequency ($\omega_\theta = \dot{\theta}$), i.e. the ratio of the two has to be a rational number, or,

$$\frac{\omega_r}{\omega_\theta} = \frac{m}{n} \ \ \leftrightarrow \ \ n\omega_r = m\omega_\theta \ , \tag{13}$$

where $m$ and $n$ are integers.

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## Problem 2

A particle moves in an attractive spherically symmetric potential

$$V(r) = -\frac{k}{r^4} \ ,$$

with $k$ constant. What is the total cross section for capture of a particle incident from infinitely far away with initial velocity $v_0$ and impact parameter $b$? Remember that

$$\sigma_{\text{capture}} = \pi b_{\text{max}}^2 \ ,$$

where $b_{\text{max}}$ is the maximum impact parameter that will result in capture.

The central force acting in this problem is of the form studied in **Problem 1** with $\beta = 4$. Therefore, we can use the general treatment of Problem 1 without having to justify all the steps again. Notice however that in this problem the potential is missing a prefactor $\frac{1}{\beta}$ and therefore you have to trace that factor of $\frac{1}{4}$ through your results for Problem 1 in order not to do mistakes.

In particular, since $\beta > 2$, we know from the sketches in part **1.c)** that the effective potential has a maximum that we can calculate to be at

$$r_0 = \left(\frac{4km}{l_0^2}\right)^{\frac{1}{2}} \ , \tag{14}$$

and the maximum value of the effective potential is,

$$V_{\text{eff}}^{\text{max}} = V_{\text{eff}}(r_0) = \frac{l_0^4}{8km^2} - \frac{kl_0^4}{16k^2m^2} = \frac{1}{16}\frac{l_0^4}{km^2} \ . \tag{15}$$

In order for the particle to be captured by the center of force, the particle has to come in with enough energy to pass the potential barrier (of the one-dimensional problem with potential $V_{\text{eff}}$). The initial energy of the particle is purely kinetic, $T_0 = \frac{1}{2}mv_0^2$, where $v_0$ is related to the angular momentum (which is a constant of motion, see part **1.a)**) by $l_0 = mv_0 b$ ($b$=impact parameter). Therefore, the particle will be captured if

$$T_0 > V_{\text{eff}}^{\text{max}} \quad \to \quad \frac{1}{2}mv_0^2 > \frac{1}{16}\frac{l_0^4}{km^2} \quad \to \quad \frac{1}{2}mv_0^2 > \frac{1}{16}\frac{m^4v_0^4b^4}{km^2} \quad \to \quad b < \left(\frac{8k}{mv_0^2}\right)^{\frac{1}{4}} \equiv b_{\text{max}} \ , \tag{16}$$

and, for a given initial velocity $v_0$ there is a maximum impact parameter for which capture is possible. The capture cross section is then given by,

$$\sigma_{\text{capture}} = \pi b_{\text{max}}^2 = \pi \left(\frac{8k}{mv_0^2}\right)^{\frac{1}{2}} \ . \tag{17}$$

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## Problem 3

Consider a system of two one-dimensional coupled oscillators described by the Lagrangian

$$L = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) - \frac{\omega_0^2}{2}(x_1^2 + x_2^2) + \alpha x_1 x_2 \ ,$$

where $\omega_0$ and $\alpha$ are real constants.

**(3.a)** Find normal frequencies and normal modes.

The normal frequencies $\omega_1$ and $\omega_2$ are solutions of the equation,

$$\det(-\omega^2 \mathbf{T} + \mathbf{V}) = 0 \longrightarrow \det \begin{pmatrix} -\omega^2 + \omega_0^2 & -\alpha \\ -\alpha & -\omega^2 + \omega_0^2 \end{pmatrix} = 0 \ ,$$

where we use that,

$$\mathbf{T} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \mathbf{V} = \begin{pmatrix} \omega_0^2 & -\alpha \\ -\alpha & \omega_0^2 \end{pmatrix} \ .$$

$\omega_1$ and $\omega_2$ are then simply given by,

$$(-\omega^2 + \omega_0^2)^2 - \alpha^2 = 0 \quad \longrightarrow \quad \omega_{1,2}^2 = \omega_0^2 \mp \alpha$$

and the corresponding normal modes $\eta_1$ and $\eta_2$ can be written as,

$$\eta_1 = a_1 e^{i\omega_1 t} \quad , \quad \eta_2 = a_2 e^{i\omega_2 t}$$

where $a_1$ and $a_2$ can be derived from the *eigenvalue-like* equations:

$$\begin{pmatrix} -\omega_1^2 + \omega_0^2 & -\alpha \\ -\alpha & -\omega_1^2 + \omega_0^2 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} = 0 \ \longrightarrow \ \begin{pmatrix} \alpha & -\alpha \\ -\alpha & \alpha \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} = 0 \ \longrightarrow \ a_{11} = a_{12}$$

$$\begin{pmatrix} -\omega_2^2 + \omega_0^2 & -\alpha \\ -\alpha & -\omega_2^2 + \omega_0^2 \end{pmatrix} \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix} = 0 \ \longrightarrow \ \begin{pmatrix} \alpha & \alpha \\ \alpha & \alpha \end{pmatrix} \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix} = 0 \ \longrightarrow \ a_{21} = -a_{22}$$

and upon normalization,

$$a_{11} = a_{12} = \frac{1}{\sqrt{2}} \ \text{ and } \ a_{21} = -a_{22} = \frac{1}{\sqrt{2}} \ .$$

**(3.b)** Write $x_1$ and $x_2$ in terms of normal modes and describe how the system oscillates in each normal mode separately.

$$
\begin{aligned}
x_1 &= a_{11}\eta_1 + a_{21}\eta_2 = \frac{1}{\sqrt{2}}e^{\omega_1 t} + \frac{1}{\sqrt{2}}e^{\omega_2 t} \quad , \\
x_2 &= a_{11}\eta_1 + a_{21}\eta_2 = \frac{1}{\sqrt{2}}e^{\omega_1 t} - \frac{1}{\sqrt{2}}e^{\omega_2 t} \quad .
\end{aligned}
$$

The displacements of the two oscillators will be given by (real part only),

$$
\begin{aligned}
x_1 &= \frac{1}{\sqrt{2}}\left[\cos(\omega_1 t) + \cos(\omega_2 t)\right] \quad , \\
x_2 &= \frac{1}{\sqrt{2}}\left[\cos(\omega_1 t) - \cos(\omega_2 t)\right] \quad .
\end{aligned}
$$

If we choose initial conditions for which only $\eta_1$ is active ($x_1(0) = x_2(0)$, $\dot{x}_1(0) = \dot{x}_2(0)$), then $x_1(t) = x_2(t)$, i.e. the two oscillators oscillate is phase with requency $\omega_1$) (symmetric mode). Vicevera, if we choose initial conditions for which only $\eta_2$ is active ($x_1(0) = -x_2(0)$, $\dot{x}_1(0) = -\dot{x}_2(0)$), then $x_1(t) = -x_2(t)$, i.e. the two oscillators oscillate with opposite phase and frequency $\omega_2$) (antisymmetric mode).

**(3.c)** Assume $\alpha \ll \omega_0^2$ (weak coupling): show that the two oscillators oscillate with the same frequency ($\omega_f = \omega_0$) and their amplitudes vary harmonically with frequency $\omega_b = \alpha/(2\omega_0)$ and opposite phase (i.e. when one of the amplitudes is minimal the other is maximal). This is the well-known phenomenon of *beats*.

In the limit $\alpha \ll \omega_0^2$ we can approximate $\omega_1$ and $\omega_2$ as follows,

$$
\begin{aligned}
\omega_1 &= \sqrt{\omega_0^2 - \alpha} \simeq \omega_0 - \frac{\alpha}{2\omega_0} \quad , \\
\omega_2 &= \sqrt{\omega_0^2 + \alpha} \simeq \omega_0 + \frac{\alpha}{2\omega_0} \quad .
\end{aligned}
$$

The displacements of the two oscillators will then be,

$$
\begin{aligned}
x_1 &\simeq \frac{1}{\sqrt{2}}\left[\cos\left(\omega_0 t - \frac{\alpha}{2\omega_0}t\right) + \cos\left(\omega_0 t + \frac{\alpha}{2\omega_0}t\right)\right] \quad , \\
x_2 &\simeq \frac{1}{\sqrt{2}}\left[\cos\left(\omega_0 t - \frac{\alpha}{2\omega_0}t\right) - \cos\left(\omega_0 t + \frac{\alpha}{2\omega_0}t\right)\right] \quad .
\end{aligned}
$$

and subsequently,

$$
\begin{aligned}
x_1 &\simeq \sqrt{2}\cos(\omega_0 t)\cos\left(\frac{\alpha}{2\omega_0}t\right) \quad , \\
x_2 &\simeq \sqrt{2}\sin(\omega_0 t)\sin\left(\frac{\alpha}{2\omega_0}t\right) \quad ,
\end{aligned}
$$

where we see that the two oscillators both oscillate with fast frequency $\omega_f = \omega_0$ while the amplitudes of their oscillations vary harmonically with slow frequency $\omega_s = \alpha/(2\omega_0)$, and opposite phase (i.e. one amplitude varies as $\cos(\omega_s)$, the other as $\sin(\omega_s)$). Every $\omega_0/\alpha$ seconds one of the two amplitudes is minimal and the other is maximal. This is the phenomenon of *beats*, which therefore happens with frequency $\alpha/\omega_0$.