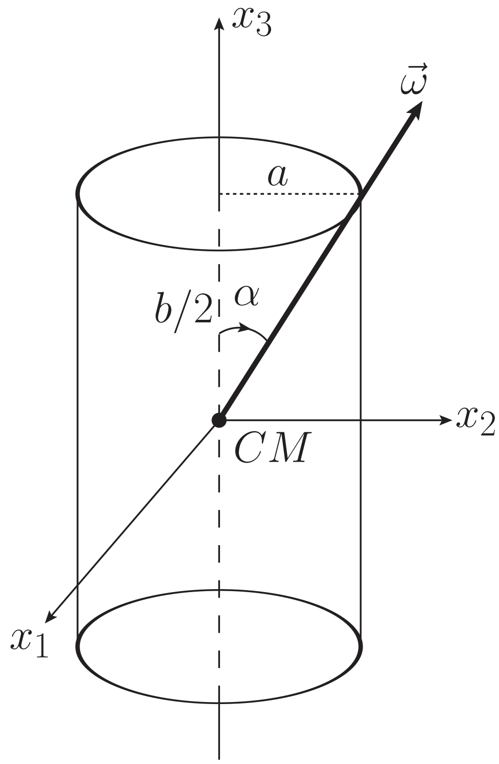
PHY 5246: Theoretical Dynamics, Fall 2015

Assignment # 10, Solutions

1 Graded Problems

Problem 1



First we calculate the moments of inertia:

$$I_1 = I_2 = m\left(\frac{a^2}{4} + \frac{b^2}{12}\right),$$

$$I_3 = \frac{ma^2}{2}.$$

(1.a)

The torque is zero! This can be seen in several ways: for instance, from the definition of the torque and the force of gravity $\mathbf{F_a} = m_a \mathbf{g}$ we can see

$$\mathbf{N} = \sum_a \mathbf{r_a} \times \mathbf{F_a} = \sum_a m_a \mathbf{r_a} \times \mathbf{g}.$$

Here $\mathbf{r_a}$ is defined with respect to the center of mass since that is the axis about which we want to calculate the torque. However, since the center of mass vector $\vec{\boldsymbol{R}}$ is defined such that

$$\mathbf{R} = \frac{\sum_a m_a \mathbf{r_a}}{\sum_a m_a} = 0,$$

we see that $\sum_a m_a \mathbf{r}_a = 0$ and so $\mathbf{N} = 0$.

(1.b)

Euler's equations are

$$\begin{cases} I_1\dot{\omega}_1 - (I_1 - I_3)\omega_2\omega_3 & = & 0 \\ I_1\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 & = & 0 \\ I_3\dot{\omega}_3 & = & 0 \end{cases}$$

$$\begin{cases} \dot{\omega}_1 - \frac{I_1 - I_3}{I_1}\omega_3\omega_2 & = & 0 \\ \dot{\omega}_2 + \frac{I_1 - I_3}{I_1}\omega_3\omega_1 & = & 0 \\ \dot{\omega}_3 & = & 0 \end{cases}$$

The last equation implies ω_3 is constant, and is just the projection of $\vec{\boldsymbol{\omega}}$ onto the x_3 axis-

$$\omega_3 = \omega\cos\alpha = \frac{\omega b}{\sqrt{b^2 + 4a^2}}.$$

Now to solve for ω_1 and ω_2, we can rewrite the first two equations as

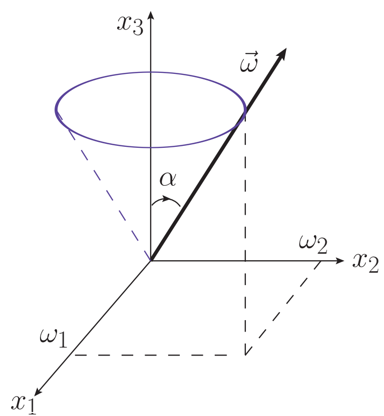
$$\begin{cases} \dot{\omega}_1 + \Omega\omega_2 &= 0 \\ \dot{\omega}_2 - \Omega\omega_1 &= 0 \end{cases}, \text{ where } \Omega = \frac{I_3 - I_1}{I_1}\omega_3.$$

Taking another derivative of the first equation with respect to time and inserting the second equation we find

$$\ddot{\omega}_1 + \Omega\dot{\omega}_2 = 0 \longrightarrow \ddot{\omega}_1 + \Omega^2\omega_1 = 0,$$

which has solution $\omega_1(t) = A\cos(\omega t + \delta)$, so that means $\omega_2(t) = A\sin(\Omega t + \delta)$. Since the phase is the same we can set $\delta = 0$ and our full solution is

$$\begin{cases} \omega_1(t) &= A\cos(\Omega t) \\ \omega_2(t) &= A\sin(\Omega t) \\ \omega_3(t) &= \frac{\omega b}{\sqrt{b^2+4a^2}} \end{cases}$$



Thus we have that $\vec{\boldsymbol{\omega}}$ precesses in a cone around x_3 with angular frequency

$$\begin{aligned} \Omega &= \frac{I_3 - I_1}{I_1}\omega_3 \\ &= \frac{\frac{ma^2}{2} - m\left(\frac{a^2}{4} + \frac{b^2}{12}\right)}{\frac{ma^2}{4} + \frac{mb^2}{12}} \frac{\omega b}{\sqrt{b^2 + 4a^2}} \\ &= \frac{3a^2 - b^2}{3a^2 + b^2}\frac{\omega b}{\sqrt{b^2 + 4a^2}} \end{aligned}$$

This also tells us what our constant A should be:

$$A = \omega_1(0) = \omega\sin\alpha = \frac{2a\omega}{\sqrt{b^2 + 4a^2}}.$$

(1.c)

The kinetic energy will be

$$T = T^{(CM)}_{trans} + T^{(about\ CM)}_{rot},$$

with

$$\begin{aligned} T^{(CM)}_{trans} &= \frac{1}{2}m(V_0 - gt)^2 \\ T^{(about\ CM)}_{rot} &= \frac{1}{2}I_1(\omega_1^2 + \omega_2^2) + \frac{1}{2}I_3\omega_3^2 \\ &= \frac{1}{2}m\left(\frac{a^2}{4} + \frac{b^2}{12}\right)A^2 + \frac{1}{2}\frac{ma^2}{2}\omega_3^2 \\ &= \frac{1}{2}m\left(\frac{a^2}{4} + \frac{b^2}{12}\right)\frac{4a^2\omega^2}{b^2 + 4a^2} + \frac{1}{2}\frac{ma^2}{2}\frac{\omega^2 b^2}{b^2 + 4a^2} \\ &= \frac{1}{2}m\frac{a^2\omega^2}{b^2 + 4a^2}\left(a^2 + \frac{b^2}{3} + \frac{b^2}{2}\right) \\ &= \frac{1}{2}\frac{ma^2\omega^2}{b^2 + 4a^2}\left(a^2 + \frac{5}{6}b^2\right). \end{aligned}$$

Adding these together we find

$$T = \frac{1}{2}m(V_0 - gt)^2 + \frac{1}{2}\frac{ma^2\omega^2}{b^2 + 4a^2}\left(a^2 + \frac{5}{6}b^2\right).$$

Problem 2

For vertical motion, we have $\theta = 0$ so that $\omega_3 = \dot{\phi} + \dot{\psi}$ and

$$p_\phi = p_\psi = I_3\omega_3.$$

The energy is simply

$$E = \frac{1}{2}I_3\omega_3^2 + Mgh,$$

and since ω_3 is constant we can define the conserved quantity

$$E' = Mgh = E - \frac{1}{2}I_3\omega_3^2.$$

In order to study the nature of the $\theta = 0$ equilibrium position, we now study how the system (top) behaves when it's displaced by an angle θ. For an arbitrary displacement about $\theta = 0$ we can then write:

$$\begin{aligned} E' &= \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2 + Mgh\cos\theta \\ &= \frac{1}{2}I_1\dot{\theta}^2 + \frac{p_\psi^2(1-\cos\theta)^2}{2I_1\sin^2\theta} + Mgh\cos\theta = Mgh, \end{aligned}$$

where I_1, I_2, and I_3 are the principal moments of inertia relative to the a body system with origin at the (fixed) tip of the top, and we have used that $I_1 = I_2$, as well as

$$\begin{aligned} \omega_1 &= \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi\ , \\ \omega_2 &= \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi\ . \end{aligned}$$

Also, we have expressed the $\dot{\phi}$ component of the angular velocity as a function of the p_ϕ and p_ψ constants of the motion, which, for the initial conditions given in this problem, satisfy $p_\phi = p_\psi = I_3\omega_3$ (see initial discussion):

$$\dot{\phi} = \frac{p_\phi - p_\psi\cos\theta}{I_1\sin^2\theta} = \frac{p_\psi(1-\cos\theta)}{I_1\sin^2\theta}\ .$$

We can then recast the energy equation in the form,

$$Mgh(1-\cos\theta) = \frac{1}{2}I_1\dot{\theta}^2 + \frac{p_\psi^2(1-\cos\theta)^2}{2I_1\sin^2\theta}\ ,$$

which is well defined even at $\theta = 0$. Using the change of variables, $z = \cos\theta$ (so that $\dot{z} = -\sin\theta\dot{\theta}$) and solving the above equation for $\dot{z}$ we get:

$$\begin{aligned} Mgh(1-z) &= \frac{1}{2}I_1\frac{\dot{z}^2}{\sin^2\theta} + \frac{p_\psi^2(1-\cos\theta)^2}{2I_1\sin^2\theta} \\ &= \frac{1}{2}I_1\frac{\dot{z}^2}{1-z^2} + \frac{p_\psi^2(1-z)^2}{2I_1(1-z^2)} \\ \to \dot{z}^2 &= \frac{(1-z)^2}{I_1^2}\left[2MghI_1(1+z) - I_3^2\omega_3^2\right]. \end{aligned}$$

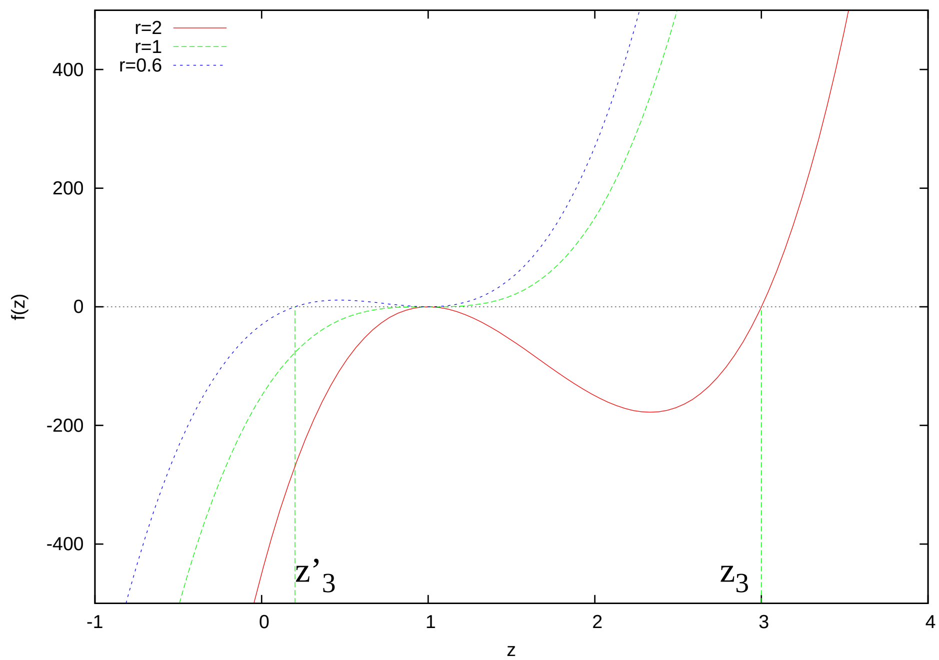


Figure 1: A plot of the function $f(z)$ given below. For simplicity we have set the parameter $\xi = 150$ and given three values of the ratio $r = \omega_3^2/\omega_c^2$. Note that $r = 1$ is the critical case.

For clarity, let us rewrite the equation in the following form,

$$\dot{z}^2\xi = (1-z)^2[(1+z) - 2r] \equiv f(z) \ ,$$

where $r = \omega_3^2/\omega_c^2$ and

$$\xi = \frac{2I_1^2}{I_3^2\omega_c^2} \ ,$$

with *critical frequency* ω_c given by,

$$\omega_c \equiv \frac{2}{I_3}\sqrt{MghI_1} \ .$$

The function $f(z)$ is plotted in the figure for three values of r: $r < 1$, $r = 1$, and $r > 1$, corresponding to three different values of ω_3: $\omega_3 < \omega_c$, $\omega_3 = \omega_c$, $\omega_3 > \omega_c$. The three zeros of the function $f(z)$ (solutions to $\dot{z}^2 = 0$) are values of z such that the motion is stationary (stable or turning point), since they correspond to $\dot{\theta} = 0$. We can see that the equation has two zeros at $z = 1$ and a third zero at $z = 2r - 1$ such that,

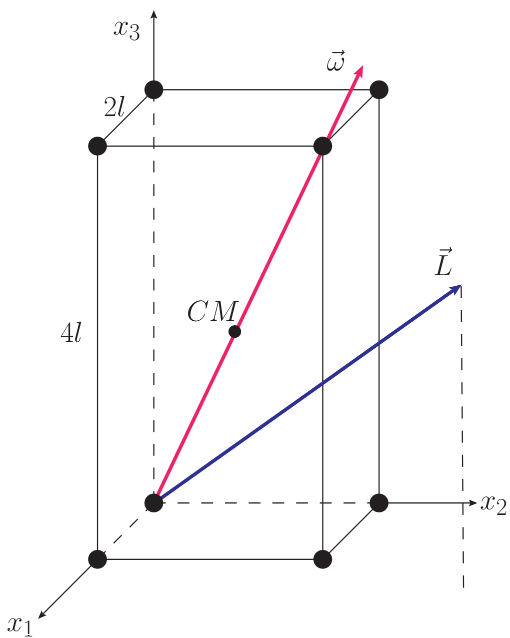
- $z > 1$ (unphysical) corresponds to $r > 1$, i.e. $\omega_3 > \omega_c$;
- $z < 1$ (physical) corresponds to $r < 1$, i.e. $\omega_3 < \omega_c$.

We can therefore describe the motion of the top as following:

- for $\omega_3 \geq \omega_c$ the top spins vertically ($\theta = 0$ is the only allowed position;
- for $\omega_3 < \omega_c$ the top spins nutating between $\theta = 0$ and $\theta = \arccos(2r-1)$.

If the top is set to spin vertically ($\theta = 0$) with $\omega_3 \geq \omega_c$ it will be stable, otherwise it will nutate. In the presence of friction, even if the top is started vertically with $\omega_3 > \omega_c$, friction will eventually reduce its angular velocity until it drops below ω_c and the top starts nutating. When friction is very low the top can spin vertically for a long time before nutations set in (case of a *sleeping top*).

Problem 3



The center of mass in the fixed coordinates is

$$CM = (l, l, 2l),$$

and for this setup we have

$$\begin{aligned}\vec{\boldsymbol{\omega}} &= \frac{1}{\sqrt{6}}(1,1,2)\cdot\omega \\ &= \omega\cdot\hat{\boldsymbol{n}}, \text{ where} \\ \hat{\boldsymbol{n}} &= \frac{1}{\sqrt{6}}(1,1,2).\end{aligned}$$

(a)

Since $\vec{\boldsymbol{\omega}}$ is constant (in both the fixed frame and the body frame since we have

$$\left(\frac{d\vec{\boldsymbol{\omega}}}{dt}\right)_{fixed} = \left(\frac{d\vec{\boldsymbol{\omega}}}{dt}\right)_{body}.$$

From the symmetry of the problem we can tell that $I_1 = I_2 \neq I_3$ (symmetric top). We can calculate these explicitly:

$$\begin{aligned}I_1 &= \sum_\alpha m_\alpha(y_\alpha^2 + z_\alpha^2) = \sum_{\alpha=1}^{8} m(y_\alpha^2 + z_\alpha^2) \\ &= 8m(l^2 + 4l^2) = 40ml^2 \\ I_2 &= \sum_\alpha m_\alpha(x_\alpha^2 + z_\alpha^2) = 40ml^2 \\ I_3 &= \sum_\alpha m_\alpha(x_\alpha^2 + y_\alpha^2) = 8m(l^2 + l^2) = 16ml^2 \\ I_{12} &= -\sum_\alpha m_\alpha x_\alpha y_\alpha = m(l^2 + l^2 - l^2 - l^2 + l^2 + l^2 - l^2 - l^2) = 0\end{aligned}$$

Where the other off-diagonal elements vanish similarly. Thus,

$$\hat{\boldsymbol{I}} = \begin{pmatrix} 40ml^2 & 0 & 0 \\ 0 & 40ml^2 & 0 \\ 0 & 0 & 16ml^2 \end{pmatrix}$$

Since the angular velocity is constant, this is not a force-free motion, since we know that the angular velocity of a symmetric top in the absence of forces precesses about the fixed direction of the angular momentum. Indeed the angular momentum is not constant in the fixed frame. We have that:

$$\begin{aligned}\vec{\boldsymbol{L}}_{body} &= I_1\omega_1\hat{\boldsymbol{e}}_1 + I_2\omega_2\hat{\boldsymbol{e}}_2 + I_3\omega_3\hat{\boldsymbol{e}}_3 \\ &= \frac{1}{\sqrt{6}}\omega ml^2(40, 40, 32) \\ &= \frac{8}{\sqrt{6}}ml^2\omega(5,5,4) = constant.\end{aligned}$$

See the figure for this vector. Therefore;

$$\left(\frac{d\vec{\boldsymbol{L}}}{dt}\right)_{fixed} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{L}},$$

and we see explicitly that $\vec{\boldsymbol{L}}$ is not constant in the fixed frame. In that this force tells us that $\vec{\boldsymbol{L}}$ precesses about the direction of $\vec{\boldsymbol{\omega}}$. We can also see that

$$\vec{\boldsymbol{L}} \cdot (\hat{\boldsymbol{e}}_3 \times \vec{\boldsymbol{\omega}}) = \vec{\boldsymbol{L}} \cdot (-\omega_2 \hat{\boldsymbol{e}}_1 + \omega_1 \hat{\boldsymbol{e}}_2) = -(I_1 - I_2)\omega_1\omega_2 = 0.$$

So, both $\vec{\boldsymbol{L}}$ and $\hat{\boldsymbol{e}}_3$ precess about the direction of $\vec{\boldsymbol{\omega}}$, keeping in the same plane with respect to each other and with respect to $\vec{\boldsymbol{\omega}}$.

(b)

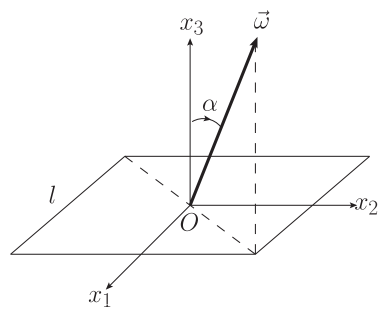
We can use Euler's equations, observing that in this frame $\dot{\omega}_1 = \dot{\omega}_2 = \dot{\omega}_3 = 0$, giving

$$\begin{aligned} N_1 &= -(I_2 - I_3)\omega_2\omega_3 \\ N_2 &= -(I_3 - I_1)\omega_3\omega_1 \\ N_3 &= -(I_1 - I_2)\omega_1\omega_2 \end{aligned}$$

From this we find

$$\begin{aligned} N_1 &= -(40 - 16)ml^2\omega^2\frac{2}{\sqrt{6}} = -8ml^2\omega^2 \\ N_2 &= -(16 - 40)ml^2\omega^2\frac{2}{\sqrt{6}} = 8ml^2\omega^2 \\ N_3 &= 0 \\ &\Rightarrow \vec{\boldsymbol{N}} = 8ml^2\omega^2(-1, 1, 0). \end{aligned}$$

Problem 4



In this problem the body axes are the principal axes, and $\vec{\boldsymbol{\omega}}$ can move in the the body fixed frame. It's easy to see that the plane is a symmetric top. Therefore, in absence of forces $\vec{\boldsymbol{L}}$ will be constant and $\vec{\boldsymbol{\omega}}$ will precess around it.

Let us calculate the moments of inertia explicitly:

$$\begin{aligned} I_1 = I_2 &= \rho \int_{-l/2}^{l/2} dx \int_{-l/2}^{l/2} dy \; x^2 = \rho \frac{1}{3}\frac{2l^3}{8}\frac{l}{2}2 \\ &= \frac{ml^2}{12} \\ I_3 &= \rho \int_{-l/2}^{l/2} dx \int_{-l/2}^{l/2} dy \; (x^2 + y^2) = \frac{ml^2}{6}. \end{aligned}$$

Now at $t = 0$,

$$\vec{\boldsymbol{\omega}} = \left(\frac{\omega \sin\alpha}{\sqrt{2}}, \frac{\omega \sin\alpha}{\sqrt{2}}, \omega\cos\alpha \right),$$

and the angular momentum is

$$\vec{\boldsymbol{L}} = (I_1\omega_1, I_2\omega_2, I_3\omega_3) = \frac{ml^2}{12}\left(\frac{\omega \sin\alpha}{\sqrt{2}}, \frac{\omega \sin\alpha}{\sqrt{2}}, 2\omega\sin\alpha \right).$$

The velocity with which $\vec{\boldsymbol{\omega}}$ precesses about $\vec{\boldsymbol{L}}$ is (see discussion in class and in the text):

$$\Omega_{pr} = \frac{L}{I_1},$$

where

$$\begin{aligned} L &= (I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)^{1/2} = \frac{ml^2\omega}{6}\left[\frac{\sin^2\alpha}{8} + \frac{\sin^2\alpha}{8} + \cos^2\alpha \right]^{1/2} \\ &= \frac{ml^2\omega}{12}(1 + 3\cos^2\alpha)^{1/2}. \end{aligned}$$

And so the frequency of precession is

$$\Omega_{pr} = \frac{(ml^2\omega/12)(1 + 3\cos^2\alpha)^{1/2}}{ml^2/12} = \omega(1 + 3\cos^2\alpha)^{1/2}.$$