PHY 5246: Theoretical Dynamics, Fall 2015

November 16^{th}, 2015
Assignment # 11, Solutions

1 Graded problems

Problem 1

1.a)

The Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) - V(r), \tag{1}$$

and the conjugate momenta are

$$\begin{aligned} p_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r} \ , \qquad (2) \\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \ , \\ p_\phi &= \frac{\partial L}{\partial \dot{\theta}} = mr^2\sin^2\theta\dot{\phi} \ . \qquad (3) \end{aligned}$$

By integrating these relations we get

$$\begin{aligned} \dot{r} &= \frac{p_r}{m} \ , \qquad (4) \\ \dot{\theta} &= \frac{p_\theta}{mr^2} \ , \\ \dot{\phi} &= \frac{p_\phi}{mr^2\sin^2\theta} \ . \end{aligned}$$

Thus the Hamiltonian is

$$H = p_i\dot{q}_i - L = \frac{1}{2m}\left[p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right] + V(r). \tag{5}$$

Using our class discussion and Goldstein § 8.1 we can find

$$\begin{aligned} \vec{\boldsymbol{a}} &= 0 \ , \qquad (6) \\ \hat{\boldsymbol{T}} &= m\begin{pmatrix} 1 & & \\ & r^2 & \\ & & r^2\sin^2\theta \end{pmatrix} \rightarrow \hat{\boldsymbol{T}}^{-1} = \frac{1}{m}\begin{pmatrix} 1 & & \\ & \frac{1}{r^2} & \\ & & \frac{1}{r^2\sin^2\theta} \end{pmatrix} \ , \\ H &= \frac{1}{2}\vec{\boldsymbol{p}}^T\hat{\boldsymbol{T}}^{-1}\vec{\boldsymbol{p}} + V \text{ with } \vec{\boldsymbol{p}} = \begin{pmatrix} p_r \\ p_\theta \\ p_\phi \end{pmatrix} \ . \end{aligned}$$

Thus we see that

$$H = T + V = E \ ,$$

as expected. Hamilton's equations of motion are

$$\begin{aligned}
&\begin{pmatrix} r \\ p_r \end{pmatrix} \quad \begin{cases} \dot{r} &= \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \\ \dot{p}_r &= -\frac{\partial H}{\partial r} = \frac{1}{mr^3}\left[p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right] - V'(r) \end{cases} \\
&\begin{pmatrix} \theta \\ p_\theta \end{pmatrix} \quad \begin{cases} \dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \\ \dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = \frac{p_\phi^2 \cos\theta}{mr^2 \sin^3\theta} \end{cases} \\
&\begin{pmatrix} \phi \\ p_\phi \end{pmatrix} \quad \begin{cases} \dot{\phi} &= \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2 \sin^3\theta} \\ \dot{p}_\phi &= -\frac{\partial H}{\partial \phi} = 0 \end{cases}
\end{aligned} \tag{7}$$

1.b)

If $p_\phi(0) = 0$ then $p_\phi(t) = 0$ for all times (since $\dot{p}_\phi = 0$ from the equations of motion). Then the equations of motion become

$$\begin{cases} \dot{r} &= \frac{p_r}{m} \\ \dot{p}_r &= \frac{p_\theta^2}{mr^3} - V'(r) \end{cases} \quad \begin{cases} \dot{\theta} &= \frac{p_\theta}{mr^2} \\ \dot{p}_\theta &= 0 \end{cases} \quad \begin{cases} \dot{\phi} &= 0 \\ \dot{p}_\phi &= 0 \end{cases} \tag{8}$$

Thus if $\phi(0) = 0 \to \phi(t) = 0$ at all times, and the motion is planar (in the $\phi = 0$ plane) as we would expect. Given the initial conditions, it will be in the $\phi = 0$ plane. The (r, p_r) and (θ, p_θ) sets of equations reduce to the usual equations for central-force motion:

$$\begin{cases} \dot{\theta} &= \frac{p_\theta}{mr^2} \longrightarrow p_\theta = mr^2\dot{\theta}, \text{ conserved.} \\ \dot{p}_\theta &= 0 \to \dot{p}_\theta = 0 \to mr(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 = ma_\theta = 0. \end{cases} \tag{9}$$

So we have that $p_\theta = l$ is the magnitude of the angular momentum, and from $\dot{p}_\theta = 0$ we get Newton's second law in the $\hat{\boldsymbol{\theta}}$ direction. For the radial part:

$$\begin{cases} \dot{r} &= \frac{p_r}{m} \\ \dot{p}_r &= \frac{p_\theta^2}{mr^3} - V'(r) = \frac{l^2}{mr^3} - V'(r) \ . \end{cases} \tag{10}$$

Taking a derivative of the first equation and plugging it into the second equation we find

$$\ddot{r} = \frac{\dot{p}_r}{m} = \frac{l^2}{m^2 r^3} - \frac{V'(r)}{m} \ . \tag{11}$$

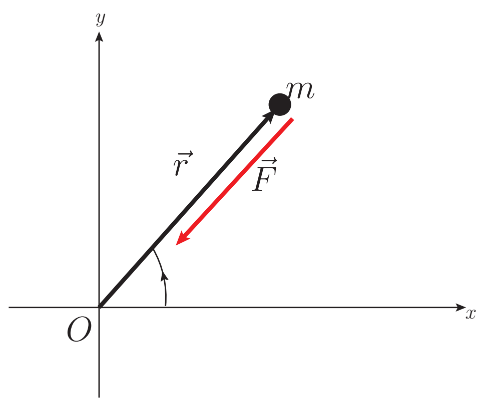
Thus we can write

$$\dot{p}_r = m\ddot{r} = \frac{l^2}{mr^3} - V'(r) = mr\dot{\theta}^2 - V'(r) \ , \tag{12}$$

$$\Longrightarrow m(\ddot{r} - r\dot{\theta}^2) = ma_r = -V'(r) = F(r) \ . \tag{13}$$

So we find this set of equations gives us Newton's 2nd law in the radial direction.

1.c)



For the specific case of $\vec{\boldsymbol{F}}(r) = -\frac{k}{r^2}\hat{\boldsymbol{r}}$, since this is a conservative force with no constraint we already know that

$$H = T + U = E \ ,$$

for constant E. From the diagram we see the coordinates are

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \implies v^2 = \dot{r}^2 + r^2\dot{\theta}^2. \tag{14}$$

The kinetic and potential energy, and the Lagrangian are

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \ , \tag{15}$$

$$U = -\int\left(-\frac{k}{r^2}\right)dr = -\frac{k}{r} \ ,$$

$$L = T - U = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r} \ .$$

For the conjugate momenta we have

$$\left\{\begin{aligned} p_r &= \tfrac{\partial L}{\partial \dot{r}} = m\dot{r} &\to \dot{r} &= \tfrac{p_r}{m} \\ p_\theta &= \tfrac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} &\to \dot{\theta} &= \tfrac{p_\theta}{mr^2} \end{aligned}\right. \tag{16}$$

Our Hamiltonian is

$$H = \dot{r}p_r + \dot{\theta}p_\theta - L = \frac{p_r^2}{2m} + \frac{p_\theta}{2mr^2} - \frac{k}{r} = T + U \ , \tag{17}$$

and E is conserved since

$$\frac{\partial H}{\partial t} = 0 \to E = \text{constant} \, . \tag{18}$$

The Hamiltonian (canonical) equations of motion are

$$\begin{aligned} \tfrac{\partial H}{\partial p_r} &= \dot{r} & \tfrac{\partial H}{\partial p_\theta} &= \dot{\theta} \\ \tfrac{\partial H}{\partial r} &= -\dot{p}_r & \tfrac{\partial H}{\partial \theta} &= -\dot{p}_\theta \end{aligned} \tag{19}$$

From the equations of motion for (θ, p_θ) we get;

$$\left\{\begin{aligned} \tfrac{\partial H}{\partial p_\theta} &= \dot{\theta} = \tfrac{p_\theta}{mr^2} \\ \tfrac{\partial H}{\partial \theta} &= -\dot{p}_\theta = 0 \end{aligned}\right. \Rightarrow p_\theta = mr^2\dot{\theta} = \text{constant}, \tag{20}$$

which shows the angular momentum is conserved. From the set of equations for (r, p_r), we find

$$\frac{\partial H}{\partial p_r} = \dot{r} = \frac{p_r}{m} \ , \tag{21}$$

$$\frac{\partial H}{\partial r} = -\dot{p}_r = -\frac{p_\theta^2}{mr^3} + \frac{k}{r^2}$$

$$\Rightarrow \dot{p}_r = m\ddot{r} = \frac{p_\theta^2}{mr^3} - \frac{k}{r^2} = mr\dot{\theta}^2 - \frac{k}{r} \ . \tag{22}$$

Problem 2

2.a)

Our particle moves in 1D, so we use one generalized coordinate x. The potential is given by integrating the force:

$$F(x,t) = \frac{k}{x^2} e^{-t/\tau} \to U(x,t) = -\int F(x,t)dx = \frac{k}{x} e^{t/\tau} + C \ , \tag{23}$$

where we assume as $x \to \infty$ that $U \to 0$ so we take $C = 0$. The kinetic energy and the Lagrangian are therefore

$$T = \frac{1}{2} m\dot{x}^2 \ , \tag{24}$$

$$\Longrightarrow L = T - U = \frac{1}{2} m\dot{x}^2 - \frac{k}{x} e^{-t/\tau} \ . \tag{25}$$

The conjugate momentum of x is

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \to \dot{x} = \frac{p_x}{m} \ . \tag{26}$$

so the Hamiltonian is

$$H = \dot{x} p_x - L = \frac{1}{2} \frac{p_x^2}{m} + \frac{k}{x} e^{-t/\tau} = T + U = E \ . \tag{27}$$

2.b)

From above we see $H = E$, since there are no constraints and U is not a function of $\dot{x}$. However, since $U = U(x,t)$ (explicitly depends on time!), the energy of the system is not conserved:

$$\frac{dE}{dt} = \frac{dH}{dt} = \frac{\partial H}{\partial t} \neq 0 \ . \tag{28}$$

Problem 3

3.a)

The magnetic field is given to us as $\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}}) = B_0 \hat{\boldsymbol{z}}$, and we can verify that the vector potential $\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = \frac{1}{2} \vec{\boldsymbol{B}} \times \vec{\boldsymbol{r}}$ satisfies $\vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}$ in the following way:

$$\begin{aligned} (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}})_i &= \epsilon_{ijk} \partial_j A_k = \epsilon_{ijk} \partial_j \left(\frac{1}{2} \epsilon_{klm} B_l x_m \right) \\ &= \frac{1}{2} \epsilon_{ijk} \epsilon_{klm} B_l \delta_{lm} \\ &= \frac{1}{2} \epsilon_{ijk} \epsilon_{klj} B_l = \frac{1}{2} 2\delta_{il} B_l = B_i \ . \end{aligned} \tag{29}$$

This implies exactly that

$$\vec{\boldsymbol{A}} = \frac{1}{2} B_0 \hat{\boldsymbol{z}} \times \vec{\boldsymbol{r}} = \frac{1}{2} B_0 (x\hat{\boldsymbol{y}} - y\hat{\boldsymbol{x}}) \ . \tag{30}$$

3.b)

The Lagrangian is

$$
\begin{aligned}
L(\vec{r}, \dot{\vec{r}}) &= \frac{1}{2}m\dot{\vec{r}}^2 + \frac{q}{c}\dot{\vec{r}}\cdot\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) \\
&= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{c}(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z) \\
&= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{qB_0}{2c}(-y\dot{x} + x\dot{y}) \ .
\end{aligned} \tag{31}
$$

The conjugate momenta are

$$
\begin{aligned}
p_x &= \frac{\partial L}{\partial \dot{x}} = m\dot{x} - \frac{qB_0}{2c}y \rightarrow \dot{x} = \frac{1}{m}\left(\frac{qB_0}{2c}y + p_x\right) \\
p_y &= \frac{\partial L}{\partial \dot{y}} = m\dot{y} + \frac{qB_0}{2c}x \rightarrow \dot{y} = \frac{1}{m}\left(-\frac{qB_0}{2c} + p_y\right) \\
p_z &= \frac{\partial L}{\partial \dot{z}} = m\dot{z} \rightarrow \dot{z} = \frac{p_z}{m} \ .
\end{aligned} \tag{32}
$$

Thus the Hamiltonian is

$$
\begin{aligned}
H &= p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - L \\
&= \frac{qB_0}{2mc}p_x y + \frac{p_x^2}{m} - \frac{qB_0}{2mc}p_y x + \frac{p_y^2}{m} + \frac{p_y^2}{m} - \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) - \frac{qB_0}{2mc}\left(-\frac{qB_0}{4c}y^2 - \frac{qB_0}{4c}x^2\right) \\
&= \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) - \frac{qB_0}{2mc}\left(-\frac{qB_0}{4c}y^2 - p_x y - \frac{qB_0}{4c}x^2 + p_y x\right) \\
&= \frac{1}{2m}\left(p_x + \frac{qB_0}{2c}y\right)^2 + \frac{1}{2m}\left(p_y - \frac{qB_0}{2c}x\right)^2 + \frac{1}{m}p_z^2 \ .
\end{aligned} \tag{33}
$$

3.c)

The mechanical momenta are

$$
\left\{
\begin{aligned}
\pi_x &= m\dot{x} = p_x + \tfrac{qB_0}{2c}y \\
\pi_y &= m\dot{y} = p_y - \tfrac{qB_0}{2c}x \\
\pi_z &= m\dot{z}
\end{aligned}
\right. \tag{34}
$$

So we have (using Landau's definition of Poisson's bracket, as also given in this problem),

$$
\begin{aligned}
\{\pi_x, \pi_y\} &= \left\{p_x + \frac{qB_0}{2c}y, p_y - \frac{qB_0}{2c}x\right\} \\
&= \{p_x, p_y\} + \frac{qB_0}{2c}\{y, p_y\} - \frac{qB_0}{2c}\{p_x, x\} - \left(\frac{qB_0}{2c}\right)^2\{y, x\} \\
&= -\left(\frac{qB_0}{2c}\right)2 = -\frac{qB_0}{c} \ ,
\end{aligned} \tag{35}
$$

where we have used that $\{p_x, p_y\} = 0$ and $\{y, x\} = 0$. Similarly we have

$$
\begin{aligned}
\{\pi_y, \pi_z\} &= \left\{p_y - \frac{qB_0}{2cx}, p_z\right\} = 0 \ , \\
\{\pi_z, \pi_x\} &= \left\{p_z, p_x + \frac{qB_0}{2c}y\right\} = 0 \ .
\end{aligned} \tag{36}
$$

(3.d)

In terms of the mechanical momenta:

$$H = \frac{\pi_x^2}{2m} + \frac{\pi_y^2}{2m} + \frac{\pi_z^2}{2m} \ . \tag{37}$$

Now using

$$\frac{d\vec{\boldsymbol{\pi}}}{dt} = \{H, \vec{\boldsymbol{\pi}}\} \ , \tag{38}$$

we get

$$\left\{ \begin{array}{lcl} \dot{\pi}_x & = & \{H, \pi_x\} = \frac{1}{2m}\{\pi_y^2, \pi_x\} = \frac{qB_0}{mc}\pi_y \ , \\ \dot{\pi}_y & = & \{H, \pi_y\} = \frac{1}{2m}\{\pi_x^2, \pi_y\} = -\frac{qB_0}{mc}\pi_x \ , \\ \dot{\pi}_z & = & \{H, \pi_z\} = 0 \ . \end{array} \right. \tag{39}$$

The last expression implies $\pi_z(t) = \text{constant} = \pi_z(0)$. Taking a derivative of the first expression and plugging in the second gives

$$\ddot{\pi}_x = \frac{qB_0}{mc}\dot{\pi}_y = -\left(\frac{qB_0}{mc}\right)^2 \pi_x = -\omega^2\pi_x \Longrightarrow \ddot{\pi}_x + \omega^2\pi_x. \tag{40}$$

The solution to this is

$$\pi_x(t) = A\cos(\omega t) + B\sin(\omega t), \tag{41}$$

while for π_y we have

$$\pi_y(t) = \frac{1}{\omega}\left(-A\omega\sin(\omega t) + B\omega\cos(\omega t)\right) = -A\sin(\omega t) + B\cos(\omega t). \tag{42}$$

With the initial conditions

$$\pi_x(0) = A, \qquad \pi_y(0) = B \ , \tag{43}$$

we can write

$$\left\{ \begin{array}{lcl} \pi_x(t) & = & \pi_x(0)\cos(\omega t) + \pi_y(0)\sin(\omega t) \\ \pi_y(t) & = & -\pi_x(0)\sin(\omega t) + \pi_y(0)\cos(\omega t) \\ \pi_z(t) & = & \pi_z(0). \end{array} \right. \tag{44}$$

From Newton's 2nd law we would get

$$m\vec{\boldsymbol{a}} = m\frac{d\vec{\boldsymbol{v}}}{dt} = \frac{d\vec{\boldsymbol{\pi}}}{dt} = \frac{q}{c}\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}} = \frac{qB_0}{mc}\vec{\boldsymbol{\pi}} \times \hat{\boldsymbol{z}} \ . \tag{45}$$

This means that $\vec{\boldsymbol{\pi}}$ moves precessing about the z-axis with frequency $\omega = \frac{qB_0}{mc}$, as we have found in the explicit expression for π_x, π_y, and π_z above. Also note that, since

$$\{H, \vec{\boldsymbol{\pi}}\} = \frac{qB_0}{mc}\vec{\boldsymbol{\pi}} \times \hat{\boldsymbol{z}} \ , \tag{46}$$

we have also found (38).