PHY 5246: Theoretical Dynamics, Fall 2015

November 23^{rd}, 2015
Assignment # 12, Solutions

1 Graded problems

Problem 1

1.a)

Given the 1-dimensional system

$$H = \frac{p^2}{2} - \frac{1}{2q^2} \ , \tag{1}$$

we want to show that

$$D = \frac{pq}{2} - Ht \tag{2}$$

is a constant of the motion. Indeed,

$$\begin{aligned} \frac{dD}{dt} &= [D, H] + \frac{\partial D}{\partial t} = \left[\frac{pq}{2} - Ht, H\right] + \frac{\partial D}{\partial t} \\ &= \left[\frac{pq}{2}, H\right] - H = \frac{1}{2}[p, H]q + \frac{1}{2}p[q, H] - H \\ &= \frac{1}{2}\left[p, -\frac{1}{2q^2}\right] q + \frac{1}{2}p\left[q, \frac{p^2}{2}\right] - H \\ &= \frac{1}{2}\left(-\frac{1}{2q^3}\right) q + \frac{1}{2}p \cdot p - \left(\frac{p^2}{2} - \frac{1}{2q^2}\right) \\ &= -\frac{1}{2q^2} + \frac{1}{2}p^2 - \frac{p^2}{2} + \frac{1}{2q^2} = 0 \ . \end{aligned} \tag{3}$$

1.b)

Now we consider a plane motion with

$$H = |\vec{\boldsymbol{p}}|^n - ar^{-n} \ , \tag{4}$$

and we want to show there is a constant of the motion

$$D = \frac{\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{r}}}{n} - Ht \ , \tag{5}$$

where $\vec{\boldsymbol{r}} \equiv (x, y, z)$ and $\vec{\boldsymbol{p}}$ is the vector of the conjugate momenta to (x, y, z).

$$\begin{aligned}
\frac{dD}{dt} &= [D, H] + \frac{\partial H}{\partial t} = \left[\frac{\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{r}}}{n} - Ht, H\right] - H \\
&= \left[\frac{\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{r}}}{n}, H\right] - H = \left[\frac{xp_x + yp_y + zp_z}{n}, |\vec{\boldsymbol{p}}|^n - \frac{a}{r^n}\right] - H \\
&= r\left[\frac{\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{r}}}{n}, |\vec{\boldsymbol{p}}|^n\right] - a\left[\frac{\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{r}}}{m}, \frac{1}{r^n}\right] - H \\
&= \frac{1}{n}\left[\sum_i p_i x_i, (\sum_j p_j^2)^{n/2}\right] - a\left[\frac{1}{n}\sum_i p_i x_i, \frac{1}{(\sum_j x_j^2)^{n/2}}\right] - H \\
&= \frac{1}{n}\sum_k p_k \frac{n}{2} 2p_k (\sum_j p_j^2)^{n/2-1} + a\frac{1}{n}\sum_k \frac{-\frac{n}{2}2x_k}{(\sum_j x_j^2)^{n/2+1}} - H \\
&= (|\vec{\boldsymbol{p}}|)^n + ar^{-n} - (|\vec{\boldsymbol{p}}|)^n - ar^{-n} \ .
\end{aligned} \tag{6}$$

1.c)

Consider the canonical transformation $Q = \lambda q$, $p = \lambda P$ but with the time dilation $t' = \lambda^2 t$. Clearly the canonical form of Hamilton equations is not preserved if t scale as in $t' = \lambda^2 t$, but we can check that the analytic form of the equations (in terms of (q, p) in one case and (Q, P) in the other) is the same. Indeed,

$$\begin{aligned}
\dot{q} &= [q, H] = \left[q, \frac{p^2}{2} - \frac{1}{2q^2}\right] = \left[q, \frac{p^2}{2}\right] = p \ , \\
\dot{p} &= [p, H] = \left[p, \frac{p^2}{2} - \frac{1}{2q^2}\right] = \left[p, -\frac{1}{2q^2}\right] = \left(-\frac{1}{q^3}\right) = -\frac{1}{q^3} \ .
\end{aligned}$$

Implement the transformation we can then derive that

$$\begin{aligned}
\dot{q} &= \frac{dq}{dt} = \frac{1}{\lambda}\frac{dQ}{dt} = \frac{1}{\lambda}\frac{dQ}{dt'}\frac{dt'}{dt} = \frac{1}{\lambda}\frac{dQ}{dt'}\lambda^2 = \lambda\frac{dQ}{dt'} \\
& \frac{dQ}{dt'} = \frac{1}{\lambda}\dot{q} = \frac{1}{\lambda}p = \frac{1}{\lambda}\lambda P = P \to \dot{Q} = P \ , \\
\dot{p} &= \frac{dp}{dt} = \lambda\frac{dP}{dt} = \lambda\frac{dP}{dt'}\frac{dt'}{dt} = \lambda\frac{dP}{dt'}\lambda^2 = \lambda^3\frac{dP}{dt'} \\
& \frac{dP}{dt'} = \frac{1}{\lambda^3}\dot{p} = -\frac{1}{\lambda^3}\frac{1}{q^3} = -\frac{1}{Q^3} \to \dot{P} = -\frac{1}{Q^3} \ ,
\end{aligned} \tag{7}$$

and prove that the form of Hamilton's equations is preserved.

Problem 2

2.a)

Given the Hamiltonian

$$H = \frac{1}{2}(\frac{1}{q^2} + p^2 q^4) \ , \tag{8}$$

the canonical equations of motion are

$$\begin{aligned}
\dot{q} &= [q, H] = \left[q, \frac{1}{2}\left(\frac{1}{q^2} + p^2 q^4\right)\right] \qquad (9)\\
&= \frac{1}{2}[q^2, p^2 q^4] = \frac{1}{2}2p[q,p]q^4 = q^4 p \ , \\
\dot{p} &= [p, H] = \left[p, \frac{1}{2}\left(\frac{1}{q^2} + p^2 q^4\right)\right] \\
&= \frac{1}{2}\left[p, \frac{1}{q^2}\right] + \frac{1}{2}[p, p^2 q^4] \\
&= -\frac{1}{2}(-2)\frac{1}{q^3} - \frac{1}{2}p^2 \cdot 4q^3 = \frac{1}{q^3} - 2p^2 q^3 \ .
\end{aligned}$$

2.b)

The canonical transformation that will make H look like the the Hamiltonian of a harmonic oscillator of position Q and conjugate momentum P is

$$\left\{ \begin{array}{lcl} P &=& \frac{\sqrt{m}}{q} \\ Q &=& \frac{1}{\sqrt{m}} p q^2 \end{array} \right. \Longleftrightarrow \left\{ \begin{array}{lcl} q &=& \sqrt{m}\frac{1}{P} \\ p &=& \frac{1}{\sqrt{m}} Q P^2 \end{array} \right. \qquad (10)$$

This transformation takes

$$H \Longrightarrow H' = \frac{1}{2m}(P^2 + m^2 Q^2)] \ . \qquad (11)$$

The generating function is

$$F_1 = -\sqrt{m}\frac{1}{q}Q \ , \qquad (12)$$

such that

$$\left\{ \begin{array}{lcl} \frac{\partial F_1}{\partial Q} &=& -P = -\sqrt{m}q \to P = \sqrt{m}q \\ \frac{\partial F_1}{\partial P} &=& p = \frac{1}{q^2}\sqrt{m}Q = \frac{p^2}{m}\sqrt{m}Q = \frac{1}{\sqrt{m}}QP^2 \end{array} \right. . \qquad (13)$$

The canonical equations in the (Q, P) variables are

$$\left\{ \begin{array}{lcl} \dot{Q} &=& \frac{\partial H'}{\partial P} = \frac{P}{m} \ , \\ \dot{P} &=& -\frac{\partial H'}{\partial Q} = -mQ \end{array} \right. . \qquad (14)$$

which, using $Q = Q(q,p)$ and $P = P(q,p)$ as given in Eq. (10), become

$$\begin{aligned}
\dot{Q} &= \frac{P}{m} \longleftrightarrow \frac{1}{\sqrt{m}}(\dot{p}q^2 + 2pq\dot{q}) = \frac{1}{\sqrt{m}}\frac{1}{q} \\
& \dot{p} = \frac{1}{q^2}\left(\frac{1}{q} - 2pq\dot{q}\right) \\
\dot{P} &= -mQ \longleftrightarrow -\sqrt{m}\frac{\dot{q}}{q^2} = -m\frac{1}{\sqrt{m}}pq^2 \\
& \dot{q} = pq^4 \rightarrow \text{ as before} \\
& \Downarrow \\
& \dot{p} = \frac{1}{q^2}\left(\frac{1}{q} - 2pq \cdot pq^4\right) \\
& \frac{1}{q^3} - 2p^2q^3 \rightarrow \text{ as before.}
\end{aligned} \tag{15}$$

Problem 3

3.a)

We can prove that the transformation

$$Q = p + iaq \ , \quad P = \frac{p - iaq}{2ia} \ , \tag{16}$$

is canonical either by showing that it preserves the form of Hamilton's equations of motion, or by verifying that the Jacobian matrix $\mathbf{M}$ of the change of variables $(q,p) \to (Q,P)$ satisfy the symplectic condition $\mathbf{J} = \mathbf{M}\mathbf{J}\mathbf{M}^{\mathbf{T}}$, or by verifying that the fundamental Poisson brackets are invariant under such transformation.

The last two proofs are very simple. Indeed, the matrix $\mathbf{M}$ is given by,

$$\mathbf{M} = \begin{pmatrix} ia & 1 \\ -\frac{1}{2} & \frac{1}{2ia} \end{pmatrix} \ , \tag{17}$$

and one can easily verify that

$$\mathbf{M}\mathbf{J}\mathbf{M}^{\mathbf{T}} = \begin{pmatrix} ia & 1 \\ -\frac{1}{2} & \frac{1}{2ia} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} ia & -\frac{1}{2} \\ 1 & \frac{1}{2ia} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mathbf{J} \ . \tag{18}$$

On the other hand, it is also easy to show that

$$\{Q,P\} = \left\{p + iaq, \frac{p - iaq}{2ia}\right\} = -\frac{1}{2}\{p,q\} + \frac{1}{2}\{q,p\} = \{q,p\} \tag{19}$$

such that the form of the fundamental Poisson brackets is preserved.

In order to prove that the canonical form of the equations of motion is preserved we need to specify the Hamiltonian. From part **1.b)** we know that the system is a one-dimensional harmonic oscillator. Therefore, in terms of (q,p) variables the Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{k}{2}q^2 \ , \tag{20}$$

and the equations of motion are

$$
\begin{aligned}
\frac{\partial H}{\partial q} &= -\dot{p} = kq \ , \\
\frac{\partial H}{\partial p} &= \dot{q} = \frac{p}{m} \ ,
\end{aligned}
\tag{21}
$$

which, upon further derivation with respect to time, can be cast in the form,

$$
\begin{aligned}
\ddot{q} + \omega^2 q = 0 \quad &\rightarrow \quad q(t) = D\cos(\omega t + \delta) \ , \\
\dot{p} = -kq \quad &\rightarrow \quad p(t) = -\frac{Dk}{\omega}\sin(\omega t + \delta) \ ,
\end{aligned}
\tag{22}
$$

where we can recognize the familiar solution for the one-dimensional harmonic oscillator in terms of two arbitrary constants (D and δ) that can be determined from the initial conditions $q(t=0) = q_0$ and $p(t=0) = p_0$.

In order to find the form of Hamilton's equations in terms of the new variables (Q, P), we need to find the transformed hamiltonian, $H'(Q, P)$, which is obtained from $H(q, p)$ by replacing $q = q(Q, P)$ and $p = p(Q, P)$, where, through some simple algebra, one see that,

$$
q = \frac{Q - 2iaP}{2ia} \ \text{ and } \ p = \frac{Q + 2iaP}{2} \ . \tag{23}
$$

Choosing $a = m\omega$ in Eq. (23), where $\omega = \sqrt{\frac{k}{m}}$, we can write

$$
H = \frac{p^2 + m^2\omega^2 q^2}{2m} = i\omega QP \equiv H' \ , \tag{24}
$$

In terms of Q and P variables, Hamilton's equations are now,

$$
\begin{aligned}
\frac{\partial H'}{\partial Q} &= -\dot{P} = i\omega P \ , \\
\frac{\partial H'}{\partial P} &= \dot{Q} = i\omega Q \ ,
\end{aligned}
\tag{25}
$$

which, upon further derivation with respect to time, can be cast in the form

$$
\begin{aligned}
\ddot{P} + \omega^2 P &= 0 \ , \\
\ddot{Q} + \omega^2 Q &= 0 \ ,
\end{aligned}
\tag{26}
$$

of the same form of Eq. (22).

3.b)

The form of the Hamiltonian in terms of Q and P and the corresponding equations of motions have been found in **1.a)**. Here we want to find the solution of Eqs. (26) and show that it corresponds to the solution of Eqs. (22), i.e. to the motion of a one-dimensional harmonic oscillator. It is indeed obvious from the form of the equations, but, to be pedantic, let us write the solution of

Eqs. (26) (in terms of two arbitrary constants A and B which could be thought as $A = Q(t = 0)$ and $B = P(t = 0)$) as

$$\begin{aligned} Q(t) &= Ae^{i\omega t} \ , \\ P(t) &= Be^{-i\omega t} \ . \end{aligned} \tag{27}$$

Substituting it into $q(t)$ of Eq. (23) we get

$$\begin{aligned} q(t) &= \frac{A}{2ia}e^{i\omega t} - Be^{-i\omega t} = \left(\frac{A}{2ia} - B\right)\cos(\omega t) + i\left(\frac{A}{2ia} + B\right)\sin(\omega t) \\ &= D\cos\delta\cos(\omega t) - D\sin\delta\sin(\omega t) = D\cos(\omega t + \delta) \ , \end{aligned} \tag{28}$$

which corresponds to $q(t)$ in Eq. (22) if we identify

$$D\cos\delta = \left(\frac{A}{2ia} - B\right) \ \text{ and } \ D\sin\delta = -i\left(\frac{A}{2ia} + B\right) \ . \tag{29}$$

Inverting these relations to get A and B,

$$\begin{aligned} A &= iaD(\cos\delta + i\sin\delta) = iaDe^{i\delta} \ , \\ B &= -\frac{D}{2}(\cos\delta - i\sin\delta) = -\frac{D}{2}e^{-i\delta} \ , \end{aligned} \tag{30}$$

and substituting them in Eq. (23) to obtain $p(t)$ gives

$$\begin{aligned} p(t) &= \frac{A}{2}e^{i\omega t} + iaBe^{-i\omega t} = \frac{1}{2}iaDe^{i\delta}e^{i\omega t} - ia\frac{D}{2}e^{-i\delta}e^{-i\omega t} \\ &= \frac{1}{2}iaD\left(e^{i(\omega t+\delta)} - e^{-i(\omega t+\delta)}\right) = -aD\sin(\omega t + \delta) \ , \end{aligned} \tag{31}$$

which corresponds to $p(t)$ of Eq. (22) for $a = m\omega = \frac{k}{\omega}$.