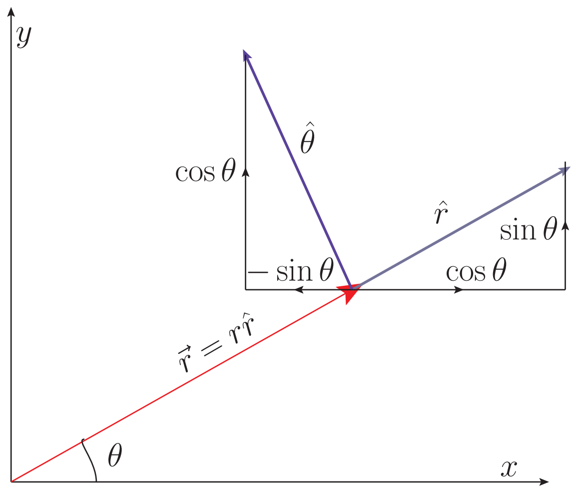
PHY 5246: Theoretical Dynamics, Fall 2015

Assignment # 1, Solutions

Problem 1



Starting from the position vector for a planar motion in polar coordinates

$$\mathbf{r} = r\hat{\mathbf{r}} \ ,$$

we can derive the velocity vector as

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta} \ ,$$

where we have used that

$$\frac{d\hat{\mathbf{r}}}{dt} = \dot{\theta}\hat{\theta} \ .$$

In the same way, we can derive the acceleration vector as

$$\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} \ ,$$

where we have used that

$$\frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{\mathbf{r}} \ .$$

Problem 2

Let us choose $\mathbf{B} = B\hat{\mathbf{y}}$ (any other choice is allowed and will lead to equivalent results, although the motion will be in different planes depending on the direction of $\mathbf{B}$). Let us write the velocity and acceleration of the pointlike charge (using cartesian coordinate) as

$$\begin{aligned} \mathbf{v} &= \dot{\mathbf{r}} = \dot{x}\hat{\mathbf{x}} + \dot{y}\hat{\mathbf{y}} + \dot{z}\hat{\mathbf{z}} \ , \\ \mathbf{a} &= \ddot{\mathbf{r}} = \ddot{x}\hat{\mathbf{x}} + \ddot{y}\hat{\mathbf{y}} + \ddot{z}\hat{\mathbf{z}} \ , \end{aligned}$$

and let us choose generic initial conditions $\mathbf{r_0} = \mathbf{r}(t=0)$ and $\mathbf{v_0} = \mathbf{v}(t=0)$ as follows

$$\begin{aligned} \mathbf{r_0} &= x_0\hat{\mathbf{x}} + y_0\hat{\mathbf{y}} + z_0\hat{\mathbf{z}} \ , \\ \mathbf{v_0} &= \dot{\mathbf{r}}_\mathbf{0} = \dot{x}_0\hat{\mathbf{x}} + \dot{y}_0\hat{\mathbf{y}} + \dot{z}_0\hat{\mathbf{z}} \ . \end{aligned} \tag{1}$$

The force acting on the pointlike charge is the Lorentz force given by (notice that there is no electric field, $\mathbf{E} = 0$)

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} = qB(\dot{x}\hat{\mathbf{z}} - \dot{z}\hat{\mathbf{x}}) \ . \tag{2}$$

The equations of motion are then

$$m\ddot{x} = -qB\dot{z} \ , \tag{3}$$

$$m\ddot{y} = 0 \ , \tag{4}$$

$$m\ddot{z} = qB\dot{x} \ . \tag{5}$$

The solution of Eq. (4) is straightforward. There is no force in the $\hat{\mathbf{y}}$ direction and the motion is the motion of a free particle, which we obtain by integrating twice with respect to time, using the initial conditions in Eq. (1):

$$y(t) = \dot{y}_0 t + y_0 \ . \tag{6}$$

The other two equations, Eqs. (3) and (5), are coupled. Differentiating Eq. (3) with respect to time and substituting Eq. (5) into Eq. (3) we obtain

$$\frac{d\ddot{x}}{dt} = -\frac{qB}{m}\ddot{z} = -\left(\frac{qB}{m}\right)^2 \dot{x} \ ,$$

which we can easily solve as a second order differential equation for $v_x = \dot{x}$ of the form

$$\ddot{v}_x + \omega^2 v_x = 0 \ \text{ with } \ \omega = \frac{qB}{m} \ .$$

The solution can be written as a linear combination of harmonic functions of the form

$$v_x(t) = A\sin(\omega t) + B\cos(\omega t) \ , \tag{7}$$

with A and B arbitrary constants to be determined imposing the initial conditions in Eq. (1). Integrating Eq. (7) one more time with respect to time we get the solution for $x(t)$ in the form

$$x(t) = -\frac{A}{\omega}\cos(\omega t) + \frac{B}{\omega}\sin(\omega t) + C \ . \tag{8}$$

In complete analogy, differentiating Eq. (5) with respect to time and substituting Eq. (3) into Eq. (5) we obtain

$$\frac{d\ddot{z}}{dt} = \frac{qB}{m}\ddot{x} = -\left(\frac{qB}{m}\right)^2 \dot{z} \ ,$$

which we can solve as a second order differential equation for $v_z = \dot{z}$ of the form

$$\ddot{v}_z + \omega^2 v_z = 0 \ \text{ with } \ \omega = \frac{qB}{m} \ .$$

obtaining

$$v_z(t) = A'\sin(\omega t) + B'\cos(\omega t) \ , \tag{9}$$

and, upon integration over time,

$$z(t) = -\frac{A'}{\omega}\cos(\omega t) + \frac{B'}{\omega}\sin(\omega t) + C' \ . \tag{10}$$

Imposing that $x(t)$ and $z(t)$ satisfy Eq. (3) (or equivalently Eq. (5)) we derive a relation among the integration constants in Eqs. (7) and (9), namely

$$A' = B \ \text{ and } \ B' = -A \ .$$

Furthermore, imposing the initial conditions in Eq. (1) we obtain

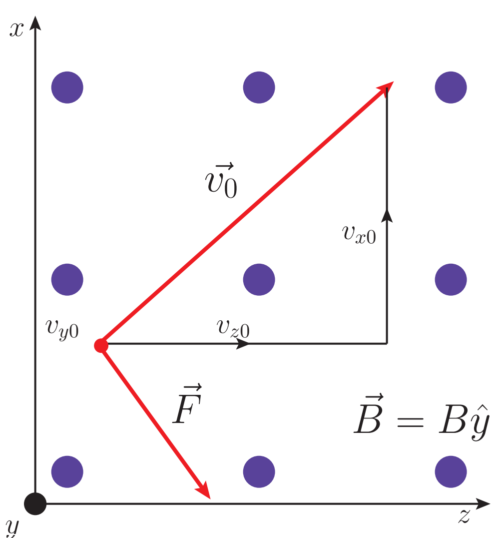
$$\begin{aligned} v_x(t=0) &= \dot{x}_0 = B \ , \qquad (11) \\ v_z(t=0) &= \dot{z}_0 = B' = -A \ , \\ x(t=0) &= x_0 = C - \frac{A}{\omega} = C + \frac{\dot{z}_0}{\omega} \ , \\ z(t=0) &= z_0 = C' - \frac{A'}{\omega} = C' - \frac{B}{\omega} = C' - \frac{\dot{x}_0}{\omega} \ . \end{aligned}$$

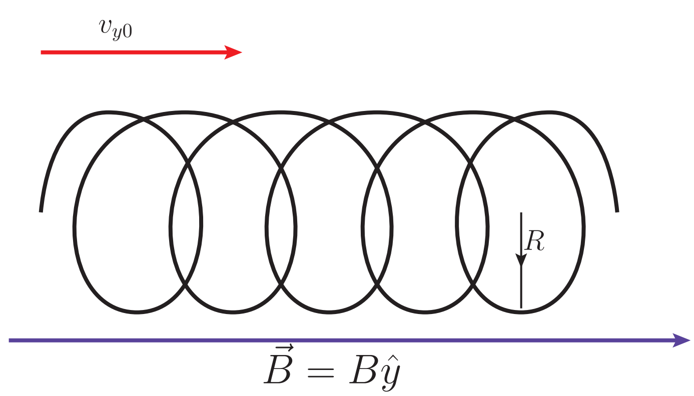
Having determined all the arbitrary constants in Eqs. (12) and (13) in terms of the initial conditions, we can finally write $x(t)$ and $z(t)$ as

$$x(t) - x_0 = \frac{\dot{z}_0}{\omega}\left(\cos(\omega t) - 1\right) + \frac{\dot{x}_0}{\omega}\sin(\omega t) \ , \tag{12}$$

$$z(t) - z_0 = \frac{\dot{x}_0}{\omega}\left(1 - \cos(\omega t)\right) + \frac{\dot{z}_0}{\omega}\sin(\omega t) \ . \tag{13}$$

From Eqs. (6), (12), and (13) we see that the pointlike charge moves in a helix with axis along the direction of the magnetic field $\mathbf{B}$ (i.e. along $\hat{\mathbf{y}}$), with radius $R = (\dot{x}_0^2 + \dot{z}_0^2)^{1/2}$.





Problem 3

If the mass m is not a function of time (m =constant) then

$$\mathbf{F}\cdot\mathbf{v} = \dot{\mathbf{p}}\cdot\mathbf{v} = m\frac{d\mathbf{v}}{dt}\cdot\mathbf{v} = \frac{d}{dt}\left(\frac{m}{2}\mathbf{v}\cdot\mathbf{v}\right) = \frac{dT}{dt} \ ,$$

where $T = mv^2/2$ is the kinetic energy of the particle.

If the mass is a function of time ($m = m(t)$) then

$$\mathbf{F} \cdot \mathbf{p} = \dot{\mathbf{p}} \cdot \mathbf{p} = \frac{1}{2}\frac{d}{dt}(\mathbf{p} \cdot \mathbf{p}) = \frac{1}{2}\frac{d}{dt}\left(m^2 \mathbf{v} \cdot \mathbf{v}\right) = \frac{d}{dt}(mT) \ .$$

Problem 4

The equations of motion of the individual particles are

$$\begin{aligned} \dot{\mathbf{p}}_\mathbf{1} &= \mathbf{F}_\mathbf{1}^{(\mathbf{e})} + \mathbf{f}_\mathbf{21} \ , \\ \dot{\mathbf{p}}_\mathbf{2} &= \mathbf{F}_\mathbf{2}^{(\mathbf{e})} + \mathbf{f}_\mathbf{12} \ , \end{aligned} \tag{14}$$

and

$$\begin{aligned} \dot{l}_\mathbf{1} &= \mathbf{r}_\mathbf{1} \times \left(\mathbf{F}_\mathbf{1}^{(\mathbf{e})} + \mathbf{f}_\mathbf{21}\right) \ , \\ \dot{l}_\mathbf{2} &= \mathbf{r}_\mathbf{2} \times \left(\mathbf{F}_\mathbf{2}^{(\mathbf{e})} + \mathbf{f}_\mathbf{12}\right) \ , \end{aligned} \tag{15}$$

Using Eq. (14) we can calculate

$$\dot{\mathbf{P}} = \dot{\mathbf{p}}_\mathbf{1} + \dot{\mathbf{p}}_\mathbf{2} = \mathbf{F}_\mathbf{1}^{(\mathbf{e})} + \mathbf{F}_\mathbf{2}^{(\mathbf{e})} + \mathbf{f}_\mathbf{21} + \mathbf{f}_\mathbf{12} = \mathbf{F}^{(\mathbf{e})} + \mathbf{f}_\mathbf{21} + \mathbf{f}_\mathbf{12} \ , \tag{16}$$

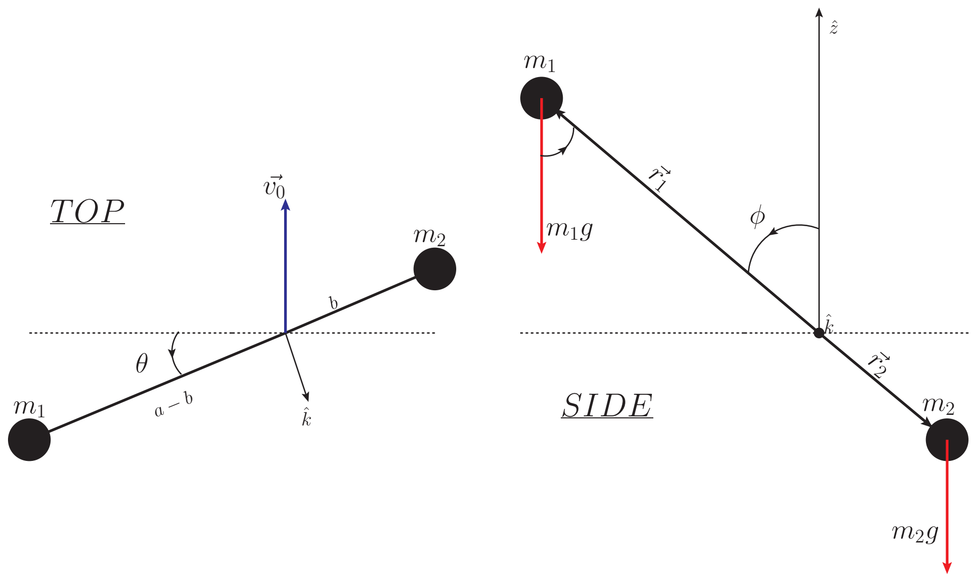
which corresponds to $\dot{\mathbf{P}} = \mathbf{F}^{(\mathbf{e})}$ only if $\mathbf{f}_\mathbf{12} = -\mathbf{f}_\mathbf{21}$ (*weak* law of action and reaction).

On the other hand, using Eq. (15) we find that

$$\ddot{\mathbf{L}} = \dot{l}_\mathbf{1} + \dot{l}_\mathbf{2} = \mathbf{r}_\mathbf{1} \times \mathbf{F}_\mathbf{1}^{(\mathbf{e})} + \mathbf{r}_\mathbf{2} \times \mathbf{F}_\mathbf{2}^{(\mathbf{e})} + (\mathbf{r}_\mathbf{1} - \mathbf{r}_\mathbf{2}) \times \mathbf{f}_\mathbf{21} \ , \tag{17}$$

where we have used the (already proved) weak law of action and reaction. The previous result corresponds to $\dot{\mathbf{L}} = \mathbf{N}^{(\mathbf{e})} = \mathbf{N}_\mathbf{1}^{(\mathbf{e})} + \mathbf{N}_\mathbf{2}^{(\mathbf{e})}$ only if $(\mathbf{r}_\mathbf{1} - \mathbf{r}_\mathbf{2}) \times \mathbf{f}_\mathbf{21} = 0$ i.e. only if $\mathbf{f}_\mathbf{21}$ (and therefore $\mathbf{f}_\mathbf{12}$) are parallel to $\mathbf{r}_\mathbf{12} = \mathbf{r}_\mathbf{1} - \mathbf{r}_\mathbf{2}$ (*strong* law of action and reaction).

Problem 5



The motion of this system is more easily understood as the superposition of the translational motion of the center of mass (CM) and the rotational motion of the two bobs with respect of to the center of mass.

The center of mass is located along the rod at a distance b from m_2 defined by the equation

$$\frac{m_2 b - m_1(a-b)}{m_1+m_2} = 0 \quad \longrightarrow \quad b = \frac{m_1}{m_1+m_2} a \ . \tag{18}$$

Since $m_2 > m_1$ the center of mass is closer to m_2, i.e. $b < a/2$. When the system is released, the center of mass is moving with an arbitrary initial velocity $\mathbf{v}_0$ and from there on its motion is governed by

$$\dot{\mathbf{P}} = \mathbf{F_1^{(e)}} + \mathbf{F_e^{(e)}} = (m_1+m_2)\mathbf{g} = M\mathbf{g} \ . \tag{19}$$

Therefore the center of mass moves like a pointlike object of mass $M = m_1 + m_2$ with initial velocity $\mathbf{v_0}$, moving under the action of the gravitational force $M\mathbf{g}$, i.e. it falls along a parabolic trajectory.

As far as the rotational motion about the center of mass goes, the system of the two masses m_1 and m_2 starts off with some given initial angular momentum $\boldsymbol{l_0}$ about the center of mass, and from there on their rotational motion is governed by

$$\begin{aligned} \dot{\mathbf{L}} &= \mathbf{N}^{(e)} = \mathbf{r_1} \times \mathbf{f_1} + \mathbf{r_2} \times \mathbf{f_2} \\ &= \left[(a-b)m_1 g \sin\phi - b m_2 g \sin\phi\right] \hat{\mathbf{k}} \\ &= \left[\left(a - \frac{m_1}{m_1+m_2}a\right) m_1 g \sin\phi - \frac{m_1 m_2}{m_1+m_2} a g \sin\phi\right] \hat{\mathbf{k}} = 0 \ , \end{aligned} \tag{20}$$

where ϕ is the angle between the plane of rotation and the direction of $\mathbf{f_1}$ (or $\mathbf{f_2}$), i.e. the vertical, while $\hat{\mathbf{k}}$ is a unit vector in the direction orthogonal to the plane of rotation of the two masses.

Since the total angular momentum of the system with respect to the center of mass is conserved, the rotational motion of the two masses happens in a constant plane. If the string does not bend, i.e. if it behaves like a rigid rod, both velocities $\mathbf{v_1}$ and $\mathbf{v_2}$ can be expressed in terms of the same angular velocity $\omega = \dot{\theta}$, where θ is the angle of rotation of the two masses (the same if the string does not bend). They can be written as $\mathbf{v_1} = b\dot{\theta}\hat{\mathbf{j}}$ and $\mathbf{v_2} = -(a-b)\dot{\theta}\hat{\mathbf{j}}$, where $\hat{\mathbf{j}}$ is a vector orthogonal to the string/rod, in the plane of rotation, and pointing in the direction of rotation.

Assuming that the string between the two masses does not bend, the tension in the string ($\mathbf{T}$) can be calculated from the radial equations of motion of the two masses written in the center of mass frame (where we use polar coordinates, and notice that the angular coordinate for the two masses, θ, is the same). Notice that these equations in any fixed frame would look like,

$$
\begin{aligned}
m_1(\ddot{r}_1 - r_1\dot{\theta}^2) &= -T - m_1\, g \cos\phi \ , &(21)\\
m_2(\ddot{r}_2 - r_2\dot{\theta}^2) &= -T + m_2\, g \cos\phi \ , &(22)
\end{aligned}
$$

but in the center of mass frame, which is accelerating with acceleration $\mathbf{g}$, the last term is missing (because we need to subtract an inertial force equal to the mass of the object times the acceleration of the frame), and the system of radial equations reads,

$$
\begin{aligned}
-m_1(a-b)\dot{\theta}^2 &= -T \ , &(23)\\
-m_2\, b\, \dot{\theta}^2 &= -T \ . &(24)
\end{aligned}
$$

where we have used that $\ddot{r}_1 = \ddot{r}_2 = 0$, since $r_1 = a - b$ and $r_2 = b$ are constant. Solving for T we get,

$$
T = \frac{m_1 m_2}{m_1 + m_2}\, a\, \dot{\theta}^2 \ , \tag{25}
$$

where we have used that $b = m_1 a/(m_1 + m_2)$.

There is another quite interesting way to look at this problem. Consider the problem as a two-body problem, i.e. as the motion of two objects under the action of a central force, directed along the line that joins the two moving objects at all times (in this case, the tension of the string). Using the appropriate set of coordinates to describe a two-body problem, i.e. making the change of variables,

$$
(\mathbf{r_1}, \mathbf{r_2}) \ \rightarrow \ (\mathbf{R}, \mathbf{r})
$$

where $\mathbf{R}$ is the position vector of the center of mass,

$$
\mathbf{R} = \frac{m_1\mathbf{r_1} + m_2\mathbf{r_2}}{m_1 + m_2}
$$

and $\mathbf{r}$ is the relative position of the two masses,

$$
\mathbf{r} = \mathbf{r_1} - \mathbf{r_2} \ ,
$$

we know that the motion of the system reduces to the motion of an object of mass equal to the total mass of the system ($M = m_1 + m_2$) located at the center of mass (i.e. with position vector $\mathbf{R}$), plus the motion of an object of reduced mass $\mu = m_1 m_2/(m_1 + m_2)$ and position vector $\mathbf{r}$, subject to the central force of the problem ($\mathbf{T}$ in our case). The equation of motion for M is then simply,

$$
M\ddot{\mathbf{R}} = M\mathbf{g} \ ,
$$

while the equation for μ is the equation of an object moving along a circular orbit of constant radius $|\mathbf{r_1} - \mathbf{r_2}| = a$, i.e.

$$\mu\, a\, \dot{\theta}^2 = T \quad \longrightarrow \quad T = \frac{m_1 m_2}{m_1 + m_2}\, a\, \dot{\theta}^2 \ .$$